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ABSTRACT

This yearbook focuses on the role of technology in school mathematics. Chapters are replete with classroom-tested ideas for using technology to teach new mathematical ideas and to teach familiar mathematical ideas better. Chapters included: (1) "Using the Graphing Calculator in the Classroom: Helping Students Solve the "Unsolvable" (Eric Milou, Edward Gambler, Todd Moyer); (2) "Graphing Changing Averages" (David Duncan, Bonnie Litwiller); (3) "Making More of an Average Lesson: Using Spreadsheets To Teach Preservice Teachers about Average" (John Baker); (4) "In the Presence of Technology, Geometry is Alive and Well--but Different" (Gina Foletta); (5) "Composing Functions Graphically on the TI-92" (Linda Iseri); (6) "Signifiers and Counterparts: Building a Framework for Analyzing Students' Use of Symbols" (Margaret Kinzel); (7) "Exploring Continued Fractions: A Technological Approach" (Tom Evitts); (8) "The Isosceles Triangle: Making Connections with the TI-92" (Karen Flanagan, Ken Kerr); and (9) "Mathematically Modeling a Traffic Intersection" (Jon Wetherbee). (ASK)

TEACHING AND LEARNING MATHEMATICS WITH TECHNOLOGY

1997 Yearbook

PENNSYLVANIA COUNCIL OF
TEACHERS OF MATHEMATICS

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Glendon W. Blume
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Co-Editors

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PREFACE

As we approach the new millennium, it seems appropriate to reflect on changes in the mathematics we teach, and in how we teach it. The single most important influence on the school mathematics of the past twenty-five years has been the coming of age of personal computing technology. The 1997 Yearbook of the Pennsylvania Council of Teachers of Mathematics, *Teaching and Learning Mathematics with Technology*, focuses on the role of technology in school mathematics. Its chapters are replete with classroom-tested ideas for using technology to teach new mathematical ideas and to teach familiar mathematical ideas better.

Technology affords teachers and students new mathematical power, and many times that new power centers on new visual images the technology affords. The first chapter, "Using the Graphing Calculator in the Classroom: Helping Students Solve the 'Unsolvable,'" by Eric Milou, Edward Gambler, and Todd Moyer, discusses the ways in which high school students approached solving a seemingly unsolvable equation by using graphing technology. At times, technology can turn problems which previously had been addressed using only symbolic manipulation into problems whose graphical representations open new ways of thinking about the concepts involved. Such is the case in David Duncan's and Bonnie Litwiller's chapter, "Graphing Changing Averages," in which the authors analyze the average number of points per game for a player given statistics related to his average. The authors use the graphing calculator to help the student visualize the graphs of many related functions simultaneously. John Baker, in his chapter, "Making More of an Average Lesson: Using Spreadsheets to Teach Preservice Teachers about Average," demonstrates how the visual representations available on a spreadsheet can be used to broaden students' concepts of average. He describes how this use of spreadsheets relates to current research on students' learning of the concept of average.

Implementation of the NCTM *Standards* is a continuing issue for today's mathematics teachers, and Gina Foletta raises an important issue about the dilemmas its implementation poses. She reminds us of "the tension between the recommendation to de-emphasize proof while at the same time to emphasize mathematical reasoning." She sheds light on the important role of technology in easing this tension in her chapter, "In the presence of technology, geometry is alive and well ... but different." The final chapter focusing on the visual role of technology is "Composing Functions Graphically on the TI-92," by Linda Iseri. Many high school and college mathematics teachers find that students struggle with the issue of the meaning of domain in composition of functions. The author of this chapter lends considerable clarity to the issue of domain by illustrating the composition graphically on a TI-92 calculator.

Much of the current and past uses of technology in the mathematics classroom have centered on graphing technology. Some have said that this emphasis has resulted in a de-emphasis on symbolic representations. Margaret Kinzel, in "Signifiers and Counterparts: Building a Framework for Analyzing Students' Use of Symbols," shares her theory about how we might think about the things that students do with algebraic notation. This theory takes on increased importance in light of recent developments in school-based technology. The past few years have seen the release of the TI-92, a user-friendly handheld computer algebra system coupled with a dynamic geometry tool. Tom Evitts, in his "Exploring Continued Fractions: A Technological Approach," illustrates the use of the symbolic manipulation capabilities of the TI-92 in the study of fractions that students seldom have the opportunity to explore. Karen Flanagan and Ken Kerr, in "The Isosceles Triangle: Making Connections with the TI-92," illustrate how tools like the TI-92 might be integrated into the teaching of geometry.

The final chapter highlights the contributions technology can make to the development of mathematical models. In his chapter, "Mathematically Modeling a Traffic Intersection," Jon Wetherbee, a middle school student, describes how he used spreadsheets and computer programming in modeling traffic flow for the purpose of simulating the effects of proposed changes to a road. Jon's chapter, as well as the previous chapters clearly illustrates the power technology offers to students and teachers of mathematics.

Glendon W. Blume
The Pennsylvania State University

M. Kathleen Heid
The Pennsylvania State University

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MATHEMATICS

USING THE GRAPHING CALCULATOR IN THE CLASSROOM: HELPING STUDENTS SOLVE THE "UNSOLVABLE"

Eric Milou
Rowan University

Edward D. Gambler
Ephrata Area Senior High School

Todd O. Moyer
Ephrata Area Senior High School

In the 21st century, technology will play an even bigger role in education in particular, and in society in general, than it does today. Educators must explore ways to use technology to promote more effective instruction. In mathematics education, the technology that affords the greatest promise is the hand-held, programmable, graphing calculator. Casio invented the first graphing calculator in 1985 and started a revolution by presenting powerful hand-held graphing capabilities to millions of mathematics students. In the intervening years, graphing calculator technology has made it possible for students to visualize mathematics on a regular basis both inside and outside of the classroom. As suggested in the National Council of Teachers of Mathematics *Curriculum and Evaluation Standards for School Mathematics* (1989), these tools have changed the very nature of the problems important to mathematics and the methods used to investigate those problems. Furthermore, the availability of graphing calculators raises questions about what is, what can be, and what should be taught in the mathematics classroom.

> Just as the four-function calculator changed the role of pencil-and-paper skills in arithmetic and the goals of elementary school mathematics, graphing and programmable calculators are forcing a serious examination of the secondary school curriculum. (Burrill, 1992, p. 15)

Graphing calculators can enhance the teaching and learning of algebra by modeling relationships graphically, numerically and symbolically. Moreover, although many equations can be solved symbolically, adding the graphing approach allows students to see connections between symbolic solutions and graphical solutions. In fact, many algebraic concepts (e.g., domain, range, inverse, solutions to equations, inequality) and their applications can be learned more effectively with a graphing calculator (Demana & Waits, 1990). Important connections between algebra and geometry can be demonstrated through the following problem.

Find all points of intersection of $y = x^4$ and $y = 3^x$.

Clearly the algebraic solution would be prohibitively complex for high school students. However, with a graphing calculator the solution of this problem is within the reach of all students.

At a large suburban Philadelphia high school, an Algebra III class of juniors and seniors was assigned this problem. The students in this class had graphing calculators (Texas Instruments TI-82's) available to them at all times. Students were placed in eight groups and given two class periods to work on the problem. One student in each group was chosen to record the group's work in a journal. Three distinct approaches to the problem follow.

Approach 1: Five of the eight groups approached the problem in a similar manner. These groups graphed both functions with a standard viewing window of [-10, 10] by [-10, 10] (See Figure 1). This can be accomplished quickly with the "zoom standard" command (Zstandard) on the TI-82. Then, using the "intersect" (Calculate 5: intersect) command, students were able to find two solutions, one at $x = -.802$, $y = .414$ and one at $x = 1.517$, $y = 5.293$. Most of the groups felt satisfied that they had all possible solutions. However, one of the groups decided to continue by changing the domain and range of the viewing window. After much debate on whether they would ever find another point of intersection, the students, using a viewing window of [5, 10] by [0, 5000], discovered another point of intersection at $x = 7.175$, $y = 2649.89$. The students in this group were very excited and exclaimed to the class that there were three solutions. Were there more?

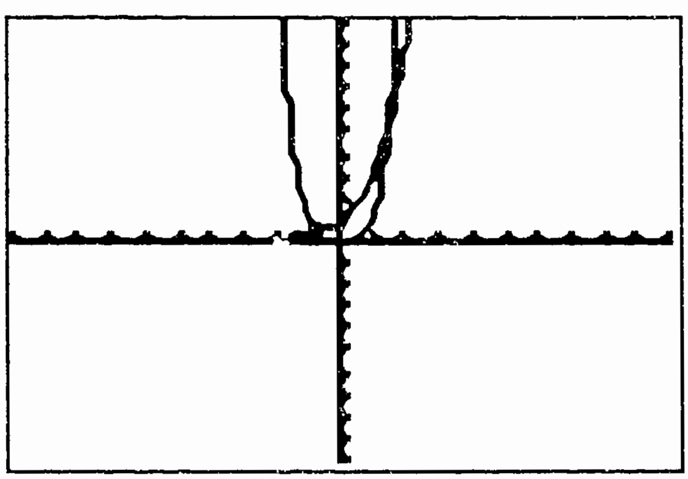

Figure 1. Graphs of $y = x^4$ and $y = 3^x$ with viewing window [-10, 10] by [-10, 10].

Approach 2: Two groups approached the problem by finding the first two points of intersection as in Approach 1 and then making a table. They used the Table Setup command to set the table to begin at $x = -2$ and the delta table equal to 1 (See Table 1).

x	$y1 = x^4$	$y2 = 3^x$
-2	16	0.11111
-1	1	0.33333
0	0	1
1	1	3
2	16	9
3	81	27
4	256	81
5	625	243
6	1296	729
7	2401	2187
8	4096	6561
9	6561	19683
10	10000	59049

Table 1. Table of values for $y = x^4$ *and* $y = 3^x$.

The students made the following observation: At $x = -2$ and $x = -1$, $y1 > y2$. However, at $x = 0$, $y1 < y2$. Therefore, there must be a point of intersection between $x = -1$ and $x = 0$. This hypothesis was confirmed by the point of intersection they had found through the graphing approach. Similarly, at $x = 1$, $y1 < y2$, but at $x = 2$, $y1 > y2$. Thus, there must be a point of intersection between $x = 1$ and $x = 2$. Again, this was confirmed by the graphs. Finally, from $x = 2$ to $x = 7$, $y1 > y2$. Then, at $x = 8$, $y1 < y2$. Thus, a third elusive point of intersection must occur between $x = 7$ and $x = 8$. [Editors' note: This approach was described in Heid & Kunkle (1988).] The students did not quit. They continued down the table and looked for any other "changes". They discovered that $y2$ continued to grew much more quickly than $y1$ and concluded that there were no more points of intersection as x grew larger. Another student checked for trends at the "negative end of x" ($y1$ approaches infinity as $y2$ approaches zero) and also concluded that there were no more points of intersection.

It is worth noting at this point that these students had no prior exposure to the Intermediate Value Theorem. This, then, is an example of the use of technology freeing students from the tedium of endless calculations, allowing them instead to analyze and interpret data generated by the technology. In this case, the calculator freed them to discover an instance of an extremely important theorem which they might otherwise have encountered only in a later mathematics course.

Approach 3: One group of students had a different approach. Instead of finding the intersection points of $y = x^4$ and $y = 3^x$, they decided to find the zeros of $y = x^4 - 3^x$. They graphed this function using a standard viewing window (See Figure 2). Immediately, they discovered that three zeros were present. Using the "root" (Calculate 5: root) command on their calculator, they found the zeros to be: $x = -.802$, $x = 1.517$, $x = 7.175$. They substituted each x back into $y = x^4$ to find the corresponding y values, $y = .414$, $y = 5.296$, and $y = 2650.25$. They changed the domain and range of their graph, but concluded that no more zeros existed.

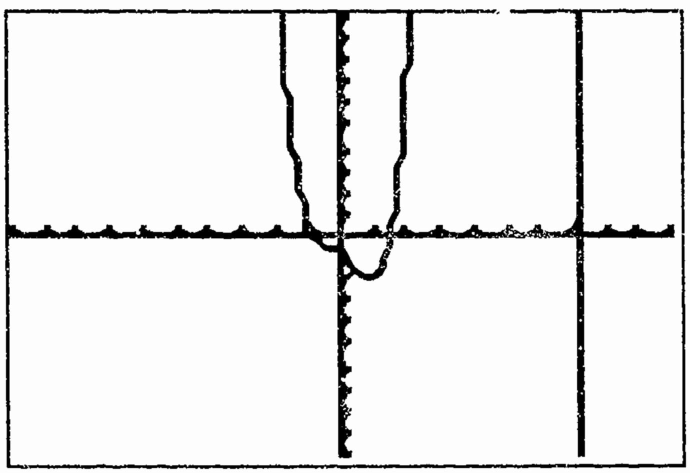

Figure 2. Graph of $y = x^4 - 3^x$ with viewing window [-10, 10] by [-10, 10].

Conclusion

When each group presented its approach, the other students were very excited to see the different methods of attacking this problem. Furthermore, it was exciting for the instructor to see Approach 2, in which the students were discovering an instance that illustrated the Intermediate Value Theorem without prior exposure to the theorem. The third approach, which tied together terminology such as zeros, roots and points of intersection, was also extremely worthwhile.

In summary, the use of the graphing calculator in the algebra classroom allows for many exciting explorations. It is a tool for connections, for cooperative work, and for discovery of mathematical principles. Moreover, this problem, which was virtually unsolvable by traditional methods, became well within the reach of every algebra student. Furthermore, the technology, when applied to the solution process, allowed for an impressive amount of real mathematical thinking and communication. This type of communication makes the core curriculum, as envisioned in the *Standards*, a real possibility. Harvey, Waits, and Demana (1995) have emphatically stated that, "falling price, easy availability, and portability of graphing calculators have provided the potential for a revolutionary impact in the teaching and learning of algebra" (p.82). This revolution can be accomplished if teachers are willing to use graphing calculators in exploration activities such as the one presented herein.

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ABOUT THE AUTHORS

Eric Milou is Assistant Professor in the Mathematics Department at Rowan University. His address is Mathematics Department, Rowan University, 201 Mullica Hill Road, Glassboro, NJ 08028-1701, and emilou@rowan.edu is his email address.

Edward D. Gambler is a mathematics teacher at Ephrata Area Senior High School. His address is Ephrata Area Senior High School, 803 Oak Boulevard, Ephrata, PA 17522

Todd O. Moyer is a mathematics teacher at Ephrata Area Senior High School. His address is Ephrata Area Senior High School, 803 Oak Boulevard, Ephrata, PA 17522

GRAPHING CHANGING AVERAGES

David R. Duncan and Bonnie H. Litwiller
University of Northern Iowa

Secondary school mathematics teachers often look for ways in which graphs can be used to represent real world situations. Graphing calculators expand the types and number of graphs that students can readily produce. The following problems concerning basketball scoring averages illustrate this.

Problem I

Suppose that an NBA basketball player, called Q, had a 20 point per game scoring average after playing in the season's first 10 games. Suppose also that for each of the remaining 72 games of the season, he scored exactly 30 points.

Note that Q's scoring average at any time is found as follows:

$$\text{Scoring average per game} = \frac{\text{Number of points scored}}{\text{Number of games played}}$$

After the first 10 games, this formula can be rewritten as $20 = \frac{\text{Number of points scored}}{10}$ or 200 = Number of points scored. Consequently, we conclude that Q has scored about 200 points in his first 10 games.

What effect does Q's enhanced performance (scoring 30 points per game for the next 72 games) have on his overall scoring average? After one additional game (11th), Q's scoring average is $\frac{200 + 30}{11} \doteq 20.9$, while after two additional games (11th and 12th), his scoring average is $\frac{200 + 2(30)}{12} \doteq 21.7$. Q's scoring average after n additional games is $\frac{200 + n(30)}{10 + n}$. We will call this Q's points per game or PPG.

Figure 1 depicts the scoring averages for $n = 0, 1, 2, ..., 72$, where n is the number of additional games after the season's first 10. Although a graph with positive integer-value inputs would be mathematically more accurate, all points on the graph are connected for ease of reading.

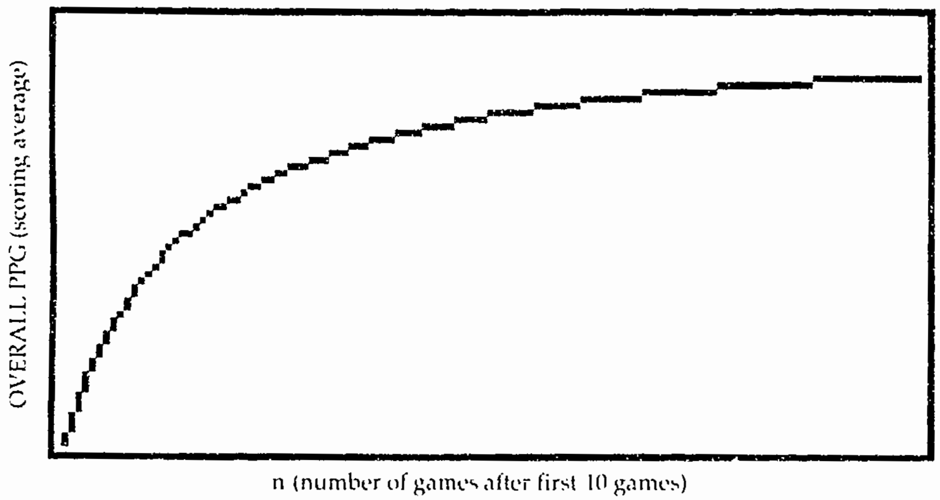

Figure 1. Scoring average for Problem 1 as a function of number of games played after the first 10 games, shown with viewing window: $0 \le n \le 72$ and $20 \le PPG \le 30$.

One should note the following three points.

1. Q's overall PPG continues to increase as the season progresses. Its largest (final) value is $\frac{200 + 30(72)}{82} \doteq 28.8$. Use the trace capability to find this point on the graph.
2. The curve is concave down, meaning that the rate of increase of PPG is slowing. The decreasing rate of increase can be verified by making a table and taking differences and second differences, or by taking the second derivative of PPG and observing its sign.
3. The graph, if continued for larger values of n, would be asymptotic to PPG = 30. How large must n be for PPG to round to 30.0? Algebraically, this means solving PPG $\ge$ 29.95.

Thus: $\frac{200 + 30n}{10 + n} \ge 29.95$

$$200 + 30n = (29.95)(10 + n)$$

$$200 + 30n \ge 299.5 + 29.95n$$

$$0.05n \ge 99.5, \text{ so}$$

$$n \ge \frac{99.5}{0.05} \text{ or } 1990.$$

The season would have to be at least 1990 + 10 or 2000 games long!

[Editors' note: Several reviewers suggested that students with access to a computer algebra system, for example, the Texas Instruments TI-92 calculator, might use the symbolic manipulation capability to solve the preceding equation. Figure 2 illustrates what students might produce on the TI-92.]

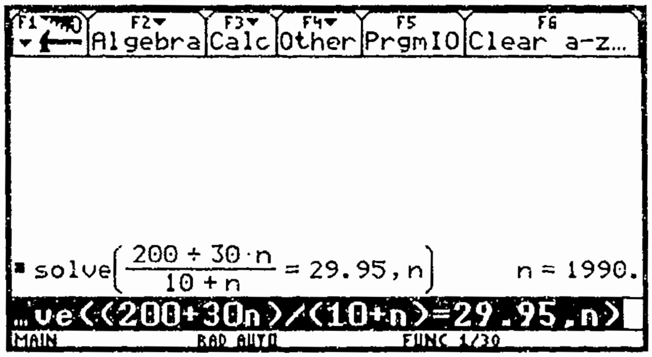

Figure 2. TI-92 solution using *solve* command.

Problem II

Suppose that Q's PPG were 30 after playing in each of the first 10 games of the season. If Q scored 20 points for each of the remaining 72 games, graph his PPG against the number of games played after the first 10.

$$PPG = \frac{300 + 20n}{10 + n}, \text{ where n} = 0, 1, 2, \ldots, 72.$$

Figure 3 depicts this situation.

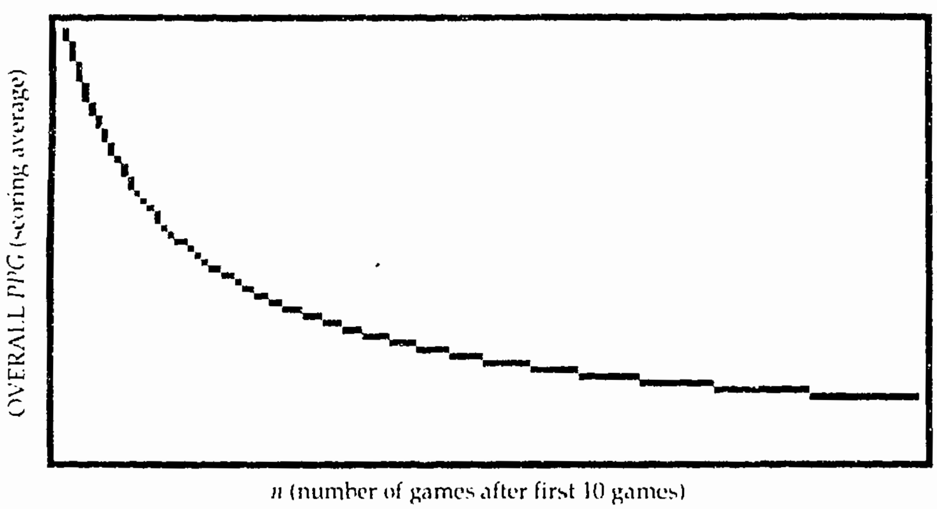

Figure 3. Scoring average for Problem II as a function of number of games played after the first 10 games, shown with viewing window: $0 \le n \le 72$ and $20 \le PPG \le 30$.

The graph in Figure 3 has "reverse" properties as compared to the graph in Figure 1. Q's PPG continues to decline for the remainder of the season, although it decreases at a slower rate as the season progresses. If the graph were extended, the curve would be asymptotic to PPG = 20. When would the PPG round to 20.0?

From these graphs students should recognize that early in the basketball season, scoring performances different from the PPG to date have a substantial influence on the overall PPG. Later in the season, the overall PPG is more difficult to change in either direction. When students recognize this, an opportunity arises to engage them in explaining why this is the case.

Problem III

Let us modify Problem I by considering a series of situations for varying numbers of initial games in which Q's PPG is 20. What effect does this have on Q's overall PPG as the season continues? For instance, suppose the initial PPG of 20 extended for 20, 30, 40, 50, 60, or 70 games.

To represent those situations on the same graph, we shall define a new set of variables. Let m represent the total number of games played at any point in the season. As in Problem I, PPG is the points per game after m games. The formulas for PPG for seven such situations are given in Table 1. In these situations Q's PPG is 20 after 10, 20, 30, 40, 50, 60, or 70 games and he scores 30 points per game thereafter to the end of the season (82 games). Figure 4 displays the graphs for these situations. Only the portion of each graph after Q's scoring increases is displayed. The straight line portions (PPG = 20) leading to the displayed curves are not shown.

These curves are similar in several ways. Each increases from left to right, although the rate of increase is slowing. If extended far enough to the right, one could see that each graph is asymptotic to PPG = 30.

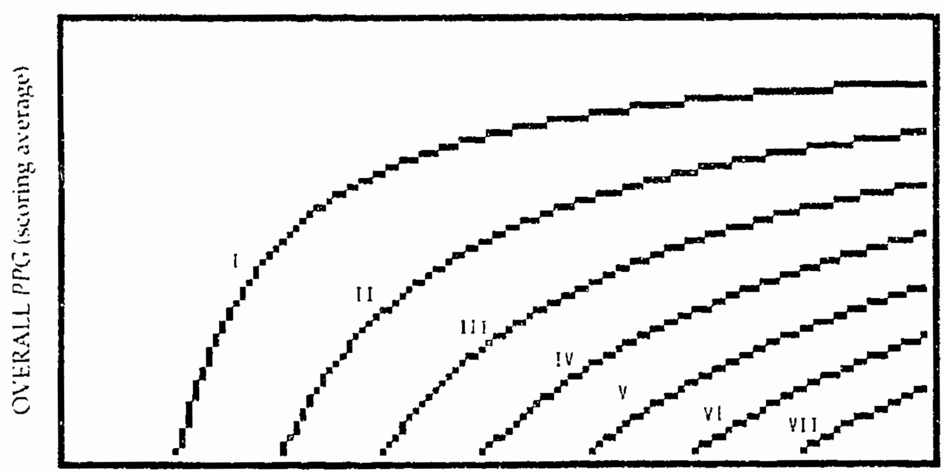

Figure 4. Scoring averages for Situations I-VII as a function of number of games played, shown with viewing window: $0 \le m \le 82$ and $20 \le PPG \le 30$.

	Number of games after which Q's PPG is 20 (he scores 30 points per game thereafter to the end of the 82-game season)	PPG
Situation I	10	$PPG = \begin{cases} 20, \text{ if } m \le 10 \\ \dfrac{200 + 30(m-10)}{m}, \text{ if } m > 10 \end{cases}$
Situation II	20	$PPG = \begin{cases} 20, \text{ if } m \le 20 \\ \dfrac{400 + 30(m-20)}{m}, \text{ if } m > 20 \end{cases}$
Situation III	30	$PPG = \begin{cases} 20, \text{ if } m \le 30 \\ \dfrac{600 + 30(m-30)}{m}, \text{ if } m > 30 \end{cases}$
Situation IV	40	$PPG = \begin{cases} 20, \text{ if } m \le 40 \\ \dfrac{800 + 30(m-40)}{m}, \text{ if } m > 40 \end{cases}$
Situation V	50	$PPG = \begin{cases} 20, \text{ if } m \le 50 \\ \dfrac{1000 + 30(m-50)}{m}, \text{ if } m > 50 \end{cases}$
Situation VI	60	$PPG = \begin{cases} 20, \text{ if } m \le 60 \\ \dfrac{1200 + 30(m-60)}{m}, \text{ if } m > 60 \end{cases}$
Situation VII	70	$PPG = \begin{cases} 20, \text{ if } m \le 70 \\ \dfrac{1400 + 30(m-70)}{m}, \text{ if } m > 70 \end{cases}$

Table 1. Formulas for PPG for seven situations.

There are also differences among these curves. The apparent curvatures of the seven graphs differ noticeably. How can this be explained?

Challenges for the Reader and for the Reader's Students

1. Replicate the situations in this article using statistics from women's professional basketball leagues. Be sure to take note of the number of games played in a regular season.
2. Find other settings for which graphing illuminates changing averages.

We have used graphing and algebra to gain insight into a statistical situation. The calculator is a tool that facilitates the graphing process and enables one to examine a variety of possible situations.

ABOUT THE AUTHORS

David R. Duncan and Bonnie H. Litwiller are Professors of Mathematics at the University of Northern Iowa. Their address is Department of Mathematics, The University of Northern Iowa, Cedar Falls, IA 50614-0506.

The Use of Technology in the Learning and Teaching of Mathematics

A Position Statement of the National Council of Teachers of Mathematics

Technology is changing the ways in which mathematics is used and is driving the creation of new fields of mathematical study. Consequently, the content of mathematics programs and the methods by which mathematics is taught and learning assessed are changing. The ability of teachers to use the tools of technology to develop, enhance, and expand students' understanding of mathematics is crucial. These tools include computers, appropriate calculators (scientific, graphing, programmable, etc.), videodisks, CD-ROM, telecommunications networks by which to access and share real-time data, and other emerging educational technologies. Exploration of the perspectives these tools provide on a wide variety of topics is required by teachers.

It is the position of the National Council of Teachers of Mathematics that the use of the tools of technology is integral to the learning and teaching of mathematics. Continual improvement is needed in mathematics curricula, instructional and assessment methods, access to hardware and software, and teacher education.

Although the nature of mathematics and societal needs are forces that drive the curriculum, the opportunities that technology presents must be reflected in the content of school mathematics. Curricular revisions allow for the de-emphasis of topics that are no longer important, the addition of topics that have acquired new importance, and the retention of topics that remain important. In the implementation of revised curricula, time and emphasis are to be allocated to the topics according to their importance in an age of increased access to technology. Instructional materials that capitalize on the power of technology must be given a high priority in their development and implementation. The thoughtful and creative use of technology can greatly improve both the quality of the curriculum and the quality of students' learning.

Teachers should plan for students' use of technology in both learning and doing mathematics. A development of ideas is to be made with the transition from concrete experiences to abstract mathematical ideas, focusing on the exploration and discovery of new mathematical concepts and problem-solving processes. Students are to learn how to use technology as a tool for processing information, visualizing and solving problems, exploring and testing conjectures, accessing data, and verifying their solutions. Students' ability to recognize when and how to use technology effectively is dependent on their continued study of appropriate mathematics content. In a mathematics setting, technology must be an instructional tool that is integrated into daily teaching practices, including the assessment of what students know and are able to do. In a mathematics class, technology ought not be the object of instruction.

(continued on page 42)

MAKING MORE OF AN AVERAGE LESSON: USING SPREADSHEETS TO TEACH PRESERVICE TEACHERS ABOUT AVERAGE

John D. Baker
Indiana University of Pennsylvania

One of the many mathematics topics preservice elementary teachers encounter in their mathematics coursework is the statistical concept of average. In my experience, many students start the semester with a fear of mathematics that leads them to hold firmly to procedures they have learned previously. The conceptual basis for their algorithms has long ago been forgotten or has been overridden by a series of steps that lack much meaning. When reintroducing my students to statistical concepts of central tendency, I find that most students understand average as "add up all the numbers and divide by how many numbers you have" and that they are not likely to evidence an understanding of the mean beyond this procedural notion. It is my job to challenge their previous learning with other ways to look at the mean and to provide a context for learning that relates to current research.

Since new teachers tend to teach as their teachers have taught them, aspiring educators need role models who, in the spirit of The National Council of Teachers of Mathematics (NCTM) *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), can also incorporate technology into their mathematics teaching in meaningful ways. I integrate a spreadsheet into a larger lesson on learning about the concept of average, and additionally, on the different ways that elementary students show their understanding of average. Through this lesson, I challenge my students to relearn the mathematics of average in ways that are concrete, meaningful, and pedagogically sound. Further, my lesson includes practical instructional ideas that teachers can use with their students.

Background for the Spreadsheet Lesson

Related Research

Mokros and Russell (1992) reported that elementary students make sense of average using five approaches:

Approach #1: Average as Mode;
Approach #2: Average as Algorithm;
Approach #3: Average as Reasonable;
Approach #4: Average as Midpoint; and
Approach #5: Average as a Mathematical Point of Balance.

Elementary students who see average as a modal value want data to exhibit one predominant value. Interpreting average as algorithm, students depend upon the school-taught algorithm for their understanding, but their grasp of the concept does not go beyond procedures. In a later report, Russell and Mokros (1996) suggested that the approaches of "average as reasonable" and "average as midpoint" are similar and reflect common sense approaches. In either case, individuals who understand average this way are comfortable with, if not insistent upon, a symmetrical distribution. In the most sophisticated approach, average as mathematical point of balance, students understand and are able to find the average as mean in a variety of real world contexts and with a variety of types of data sets.

College students whom I have observed show a definite preference for understanding average as an algorithm, but they often have flexibility to apply situation-based estimates that are reasonable. When my students are forced to choose values that have a given mean, they most typically select data that are symmetrical. Many college students seem to have achieved a level of understanding beyond that of average as mode, algorithm, and median, but do not fully understand the mean as a balance point. With a view of average as a balance point, students can manipulate many visual representations of the same mean. I have designed the following lesson to elevate students' understandings so that they view average as a balance point.

Rationale for using a Spreadsheet

Spreadsheets offer a convenient tool for developing applications that help students focus their attention on concepts, so I decided to design a spreadsheet for my lesson on average as balance point. In general terms, I developed my spreadsheet to address the following goals:

1. To relieve students of computational burdens involved in calculating the mean so that the concept receives the central focus;
2. To make quick and accurate drawings so graphs can be accurately interpreted and inaccurate drawings can be avoided;
3. To show multiple representations (via tables, calculations, and graphs) that help students build connections; and
4. To provide interactive mechanisms for student exploration so that mistakes, as revealed by the calculation of average in the spreadsheet, can easily be corrected by making changes to it.

I addressed the first goal, automatic calculation of the mean, by programming spreadsheet table entries to perform the needed computation so that only the result of the formula was displayed. The other three goals are addressed by tables and graphs that are linked to the numerical spreadsheet. Each table is accompanied by a graph, and, with each new value entered into a table, the graph and mean are instantly updated. The user can use the spreadsheet to assess the impact of incremental changes in data on the value of the mean and on the graph.

Activities to Develop the Concept of Average

Before I introduce the spreadsheet, my classes learn about average from three activities designed for exploring the concept of average and for developing a broader perspective of how average works in different situations. In the first activity, small groups are challenged to find their mean height without paper and pencil. College students use a variety of strategies: some groups try to use the formula with mental arithmetic, others use nonstandard measures and then indicate what is reasonable, and others total the heights as one long measurement and then divide by the number of people. This latter strategy affords me the opportunity to help students make a connection between the paper and pencil algorithm and their conceptual understanding of average. With this initial activity, students take their first step in ceasing to rely on the paper and pencil algorithm.

For the second activity, I give students written problems to solve. A sample problem is:

> What must the fifth person weigh if the average weight of 5 people is 156 pounds, one person weighs 140 pounds, and the average weight of the 3 others is 180 pounds?

Problem-solving situations like these force students to work backwards in finding the mean. That is, the average is given and values must be found that fit. In this way, students are challenged to think beyond the "add up all the numbers and divide by how many numbers there are" strategy.

The third activity involves using a meter stick and weights to explore average. First I balance a meter stick at 50 centimeters. Then I place several large paper clips at various strategically selected centimeter markings on the meter stick to throw it off balance. Then, students discuss placements of one additional paper clip, usually with trial and error, to restrike a balance. After a solution is found, I use the centimeter measurements of the paper clips and the computational algorithm for the mean to show that the balance point is again 50 centimeters. My students are amazed to find that mathematics can model the real world.

The Spreadsheet Lesson

The spreadsheet lesson reinforces knowledge gained from the activities just described while providing the opportunity to discuss two other ideas: a) the five approaches that elementary students use to show their understanding of average and b) the advantages of using technology. Typically, my lesson format is teacher demonstration and class discussion. The class discussion allows me to provide visual representations of the mean while focusing students' attention on the connections to research and the advantages of using computer tools. Since the context employed in this spreadsheet is familiar to school children, it could easily be adapted for use in the future classrooms of my college students. Elementary or middle school students could first be introduced to the basic spreadsheet as a demonstration and then they could be allowed to explore on their own. I envision a teacher assessing elementary students' concepts of average from spreadsheet printouts the students have produced.

To introduce my lesson, I describe the following problem situation to the class: I am trying to find sets of 10 families so that the mean number of children is 3. Their task is to determine how many children (up to 8) are in each family. We tackle the following scenarios, one scenario at a time:

Scenario #1: How many children are in each family if all of the families have the same number of children? The mean must be 3.

Scenario #2: If 3 of the 10 families each have 2 children and 3 other families each have 4 children, how many children will each of the remaining 4 families have so that the mean number of children is 3? Figure 2 shows what this scenario initially looks like on the spreadsheet before a solution is found.

Scenario #3: If none of the families has exactly 3 children, how many children could each of the families have?

Scenario #4: Only one family has 3 children. How many children do the other 9 families have? Figure 3 shows what this scenario and Scenario #5 initially looks like on the spreadsheet before solutions are found.

Scenario #5: Two families have 8 children. How many children do each of the rest of the 8 families have?

The scenarios listed above relate directly to the five approaches to understanding average found in research (Russell & Mokros, 1996; Mokros & Russell, 1992). Scenario #1 has only one answer and is easily answered and understood by all approaches to average. Figure 1 shows the table and spreadsheet after they have been filled in. With this first scenario, I usually ask students what they see as advantages to using technology and then talk about the advantages related to research I discussed earlier in this article.

Make 3 be the mean number of children in 10 families —

Number of children in a family:

1	2	3	4	5	6	7	8	Average
		10						3

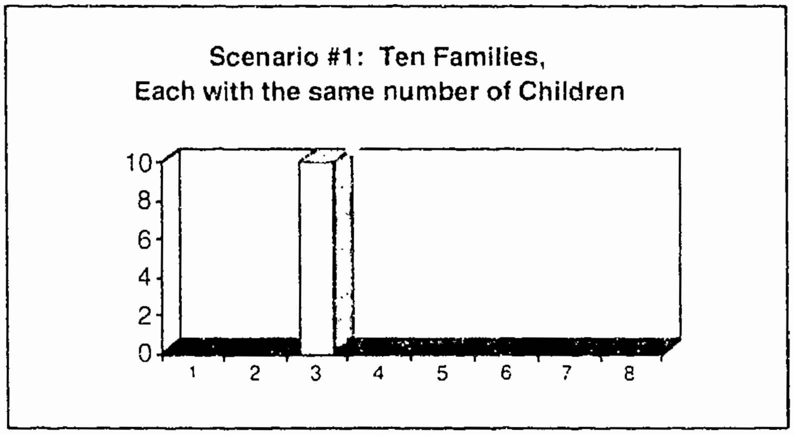

Figure 1. Spreadsheet scenario understood by all approaches.

After the first scenario, the class is shown Scenario #2 as pictured in Figure 2. The answer is easily completed by those who approach average as a mode and as a median. In the former approach, the table is filled in with 4 families with 3 children and in the latter, a symmetrical table and graph are created. This affords an opportunity to identify my students' approaches to average. Scenario #3 is similar to the second scenario, except that no families can have 3 children and no family sizes are prescribed. Students who approach average as mode will have trouble with this scenario, but can be put in a position to explore distributions, checking their work by looking at the calculated average. I like to call on a student who answered the second scenario with an average as mode approach to see if they can stretch their understanding to an approach as median.

Number of children in a family:

1	2	3	4	5	6	7	8	Average
	3		3					3

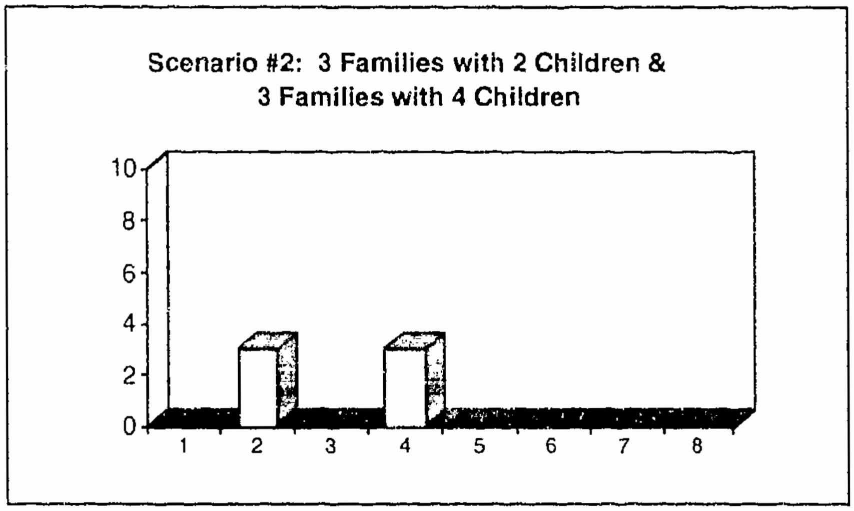

Figure 2. Spreadsheet scenario for average as mode and average as median.

Scenarios #4 and #5 are predicted to be difficult for students who do not approach average as a balance point. These students are able to find combinations of children in families that average 3, and even if they make mistakes, they can easily make changes to the table to fit the scenario. I find that with a guess-and-check strategy, college students can learn to fill in the table correctly (See Figure 3.). When students are able to explore this spreadsheet, they find different answers and consult with one another to aid in their appreciation and understanding of the questions posed.

As a follow-up to the last two scenarios, I help students understand the approach to average as a balance point by introducing them to unpacking (Russell & Mokros, 1996). This is done by students starting with the answer for Scenario #1 and, family by family, redistributing the children in the 10 families to maintain an average of 3. The final redistribution is the solution found for Scenario #4 and Scenario #5. At this point, I also like to relate the different scenarios to the algorithm by showing the students the calculations with the different scenarios that have been created. In this way, I can reinforce the notion that a total of 30 children must be accounted for in each of the scenarios.

Number of children in a family:

1	2	3	4	5	6	7	8	Average
		1						3
							2	8

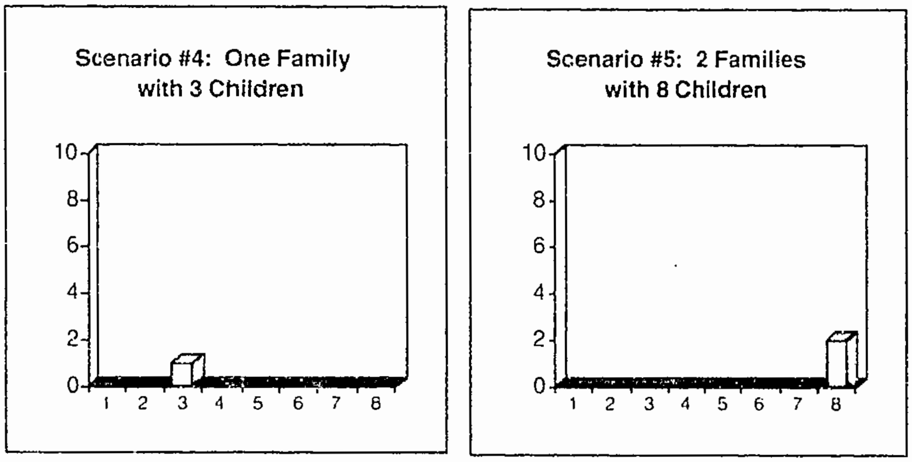

Figure 3. Spreadsheet scenarios for average as balance point.

The grand finale of the lesson is to show all 5 graphical representations of different distributions on the same screen. One way that the five graphs could look is depicted in Figure 4. After prompting students for their impressions of the graphs, the final activity is to ask students to reflect on what they have learned about their personal approach to average and their understanding of how technology can enhance a lesson to make it more than average.

Conclusion

I observe informally and on class tests that my students have success in increasing their understandings of average and their flexibility with problem-solving tasks. One student's test answer illustrates the emergence of understandings that I observe. The test required students to find one value that, when combined with other given values, produces a given average. One student successfully answered and drew pictures of a balance made from the number line. This student found the correct solution by drawing the distances from the given values to the given mean. The value required to balance the distances was the solution.

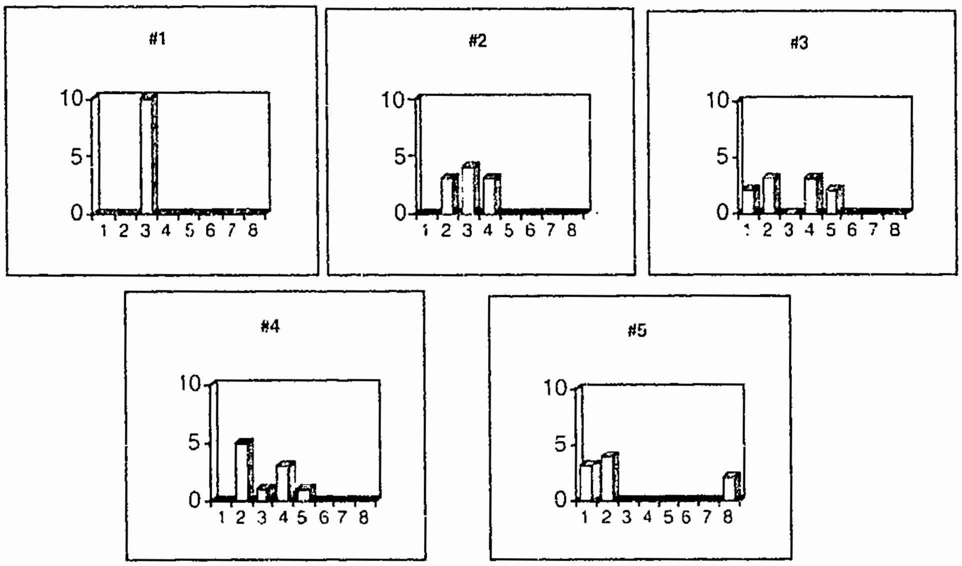

Figure 4. Multiple instances of a graphical representation displayed at the same time.

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ABOUT THE AUTHOR

John D. Baker is Assistant Professor in the Department of Mathematics at Indiana University of Pennsylvania. His address is Department of Mathematics, Indiana University of Pennsylvania, Indiana, PA 15705.

IN THE PRESENCE OF TECHNOLOGY, GEOMETRY IS ALIVE AND WELL ... BUT DIFFERENT

Gina M. Foletta
University of Northern Kentucky

As the National Council of Teachers of Mathematics (NCTM) *Standards* (NCTM, 1989,1991,1995) come under scrutiny, those of us involved in teacher education or the teaching of mathematics may reflect upon our own courses and the impact the *Standards* have had over the years. I am reminded of the tension between the recommendation to de-emphasize proof while at the same time to emphasize mathematical reasoning. I have seen courses in geometry move from including formal, axiomatic proof to abandoning proof under the guise of informal geometry to incorporating informal investigations in the hopes of moving toward some type of justification. Has the increased role of experimental activities and conjectures in school mathematics diminished the need for or value of careful reasoning and justification?

Voices of Concern

Cuoco (1995) voices grave concern about the current direction of reform in mathematics education. He expresses concern over essential features of mathematics missing in many of the reform curricula designed for secondary mathematics. In particular, Cuoco sees an absence in the manipulation of symbols as tools for thinking as well as establishing logical connections through proof and explanation. Jones (1995) claims that few teachers have experienced mathematics as investigating, patterning, abstracting, and generalizing. Fey's (1993) summary of several reports at ICME-7 indicates that dynamic geometric construction tools seem to help students focus and understand invariant properties. In addition, these tools may also help students analyze complex problems not easily accessible with only paper and pencil.

Foletta (1994) and Zbiek (1992) document that secondary students and preservice secondary teachers tend to generalize based on one computer-generated image. Martin and Harel (1989) report that preservice elementary teachers view proof as "what convinces me." Mathematicians arrive at truth by methods that are intuitive or empirical in nature (Lakatos, 1976). One value for students to engage in logical reasoning or establishing proofs is the potential for making logical connections to other mathematical concepts and developing new insights.

The Setting

The next section presents one example of some prospective middle-grades teachers' thinking and justifications while investigating and reasoning about geometric ideas with the aid of *The Geometer's Sketchpad* (Jackiw, 1995). This example comes from my geometry course for prospective middle-grades teachers (hereinafter referred to as "students") that is the first experience these students have using a dynamic geometry tool in a college mathematics course. For many of the elementary education majors pursuing an endorsement in middle-grades mathematics, this is the last mathematics course they take prior to student teaching.

I have several purposes for integrating technology in this course. The use of an electronic construction tool like *The Geometer's Sketchpad* can facilitate the exploration of geometric ideas. The tool permits students to generate data accurately and quickly in a relatively short period of time. This has the potential to allow students to focus on the inductive reasoning process of looking for patterns, conjecturing, and verifying or refuting their conjectures. The experience presents the computer as one of several tools for doing mathematics. In addition, using technology in a laboratory environment encourages communication about mathematical ideas and the making of mathematical connections.

Midpoint Polygons

The open-ended problem consists of several parts. Students are to construct regular and non-regular polygons (triangle, quadrilateral, pentagon, and hexagon) and their respective midpoint polygons by connecting the midpoints of the segments of the original polygon. Then after examining several examples, they compare each polygon with its midpoint polygon. From numerical and visual data, the students attempt to determine what relationships, if any, exist between the original polygon and its midpoint polygon. Zbiek (1996) has developed this problem for the pentagon, and this chapter extends the problem to other polygons.

Figure 1 displays a summary of typical students' conjectures. For example, most students observe that in a regular triangle the midpoint triangle is congruent to the three outer triangles (the sides of the midpoint triangle partition the interior of the original triangle into four triangular regions). This is usually the foundation for their argument in support of their conjecture that the area of the midpoint triangle of a regular (i.e., equilateral) triangle is one-fourth the area of the regular triangle (denoted by "MP = 1/4 P" in Figure 1). Students also conjecture that the midpoint polygon of a regular polygon is similar to the original polygon and that the midpoint polygon of a non-regular triangle is similar to the original triangle. Most students are able to give logical arguments for the area relationships of non-regular triangles and regular polygons (except for regular pentagons) by subdividing the polygon into congruent triangles. Few students are successful with non-regular quadrilaterals. A closer look at one group's unexpected conclusions and reasoning about non-regular triangles and pentagons gives important insights into the nature of justification and proof in this technological environment.

n	Regular Polygon	Non-regular Polygon
3	MP = 1/4 P	MP = 1/4 P
4	MP = 1/2 P	MP = 1/2 P
5	MP ~ .65P	No Pattern
6	MP ~ .75P	No Pattern

MP signifies the area of the midpoint polygon and P signifies the area of the original polygon.

Figure 1. Students' conjectures about the ratio of the area of a midpoint polygon to the area of its original polygon.

Non-regular Triangles

The group's conjectures about non-regular triangles drew on reasoning about geometric transformations rather than on algebraic reasoning to justify the area relationships. They used results from the first conjecture as part of their justification for the second conjecture.

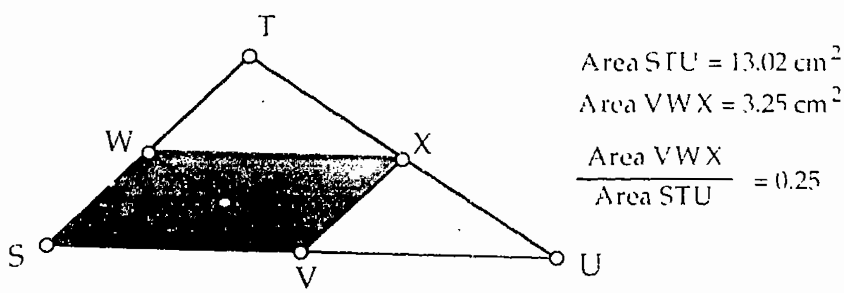
> Conjecture 1: Since the midpoint triangle of any triangle is "one half of a parallelogram" [sic], the outer triangle can be rotated to exactly cover the midpoint triangle.
>
> Justification: A parallelogram has no lines of reflection, but it does have one non-trivial rotation symmetry. The turn center is at the midpoint of the segment which forms the diagonal of the parallelogram. (For example, parallelogram SVXW in Figure 2 can be rotated 180° about point L in ΔSVW.) Therefore, each outer triangle can be rotated 180° to cover the midpoint triangle exactly.

The students then used their Conjecture 1 as part of the justification for their next conjecture.

> Conjecture 2: The midpoint triangle formed by joining the midpoints of the segments of a non-regular triangle will always cover exactly 1/4 the "a a" [sic] of the original triangle.

Justification: Since each midpoint segment is parallel to and 1/2 the length of the third original side, the midpoint triangle will always be 1/2 of a parallelogram. Any of the outer triangles can be rotated 180° to exactly cover the midpoint triangle. Since the resulting stack of triangles is exactly 4 deep of the midpoint triangle, we can say with certainty that the area of the midpoint triangle is always equal to 1/4 the area of the original triangle.

The students were describing how each outer triangle rotated about the midpoint of one side fit exactly upon the midpoint triangle; for example, in Figure 2 ΔWSV is congruent to ΔVXW.



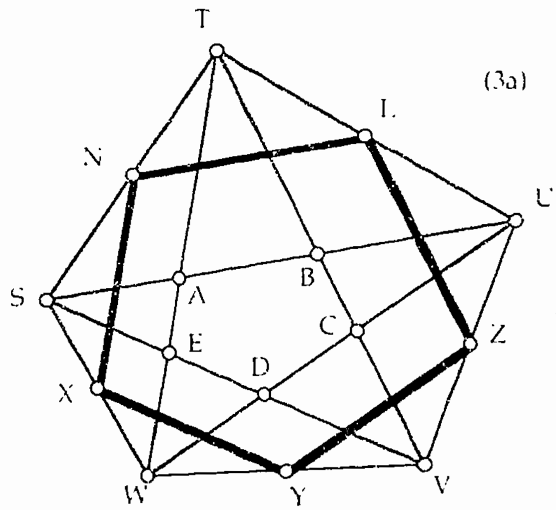
Area(Polygon VWX)/Area(Polygon STU)	0.25	0.25	0.25	0.25	0.25
Area(Polygon STU)	7.41	9.29	10.80	12.75	13.02
Area(Polygon VWX)	1.85	2.32	2.70	3.19	3.25

Figure 2. Measurements for ΔSTU and its midpoint ΔXVW generated by moving vertices of ΔSTU.

Non-regular Pentagons

This group of students went beyond the conclusion that fc· non-regular pentagons no pattern exists for the ratio of the area of the midpoint pentagon to the area of the original pentagon. They conjectured and then argued that the ratio of the areas must be greater than one-half. Their justification was based on reflecting each of the five outer triangles in each respective side of the midpoint pentagon. For example, Figure 3b shows ΔXYW reflected in side XY. The group's conclusion was based on the observation that the midpoint pentagon is not completely covered after all five outer triangles were reflected in a similar manner. The students examined several instances by dragging vertices of the pentagon and they observed that there always exists a pentagonal "hole" ABCDE in the interior of the original pentagon. However, they overlooked the fact that the reflected triangles may sometimes overlap.

In spite of the group's summary that as the number of sides of the original polygon increases the ratio of the areas also increases, they seemed to ignore the pattern that the ratio of the area of the midpoint pentagon to the area of its original pentagon must then be greater than that of the preceding quadrilateral case; that is, greater than one-half. In a similar activity, Zbiek (1996) discusses multiple approaches and the importance of reasoning within and beyond technology in a pentagon investigation.



Area(STUVW) =33.857 cm^2

Area (LNXYZ) =22.052 cm^2

$$\frac{\text{Area (LNXYZ)}}{\text{Area (STUVW)}} = 0.651$$

Area(LNXYZ)/Area(STUVW)	0.648	0.651	0.665	0.644	0.650
Area(LNXYZ)	23.481	20.726	14.222	28.230	37.387
Area(STUVW)	36.253	31.845	21.397	43.867	57.480

(3b)

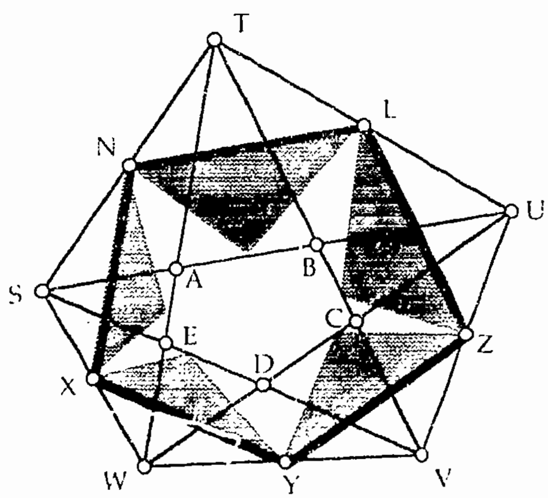


Figure 3. Lopsided five-pointed star produced by reflecting each outer triangle of pentagon STUVW.

Conclusions

As my students work on open-ended investigations with *The Geometer's Sketchpad*, they tend to view geometric sketches in complex ways. They analyze polygons as the union of parallelograms and parallelograms as the union of two congruent triangles. The students approach the investigation using the notions of rotation and reflection as rigid motions and elementary ideas of compactness in their efforts to determine a covering for the original pentagon. These examples present some evidence that technology can aid students in their exploration of mathematics and that reasoning is an important part of investigations. As inductive and deductive reasoning becomes a regular part of their mathematics experience, I hear positive statements about justification rather than the once frequently asked question, "How many conjectures should I have?"

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COMPOSING FUNCTIONS GRAPHICALLY ON THE TI-92

Linda Iseri
The Pennsylvania State University

A typical approach to teaching composition of functions is primarily symbolic in nature. Students see exercises such as:

Define $f(x) = x^3 + 2x^2 - x$ and $g(x) = x^2 - 1$ and evaluate each of the following: $f(g(2))$ $g(f(2))$ $f(g(x))$ $g(f(x))$.

We would probably expect students to produce something like:

$$f(2) = 2^3 + 2*2^2 - 2 = 14 \quad \text{so } g(f(2)) = g(14) = 14^2 - 1 = 195.$$

For a composed function rule, we would want students to evaluate the outside function at the inside function and then manipulate the symbols to some appropriate form. Students might write:

$$g(f(x)) = g(x^3 + 2x^2 - x) = (x^3 + 2x^2 - x)^2 - 1$$

and end up with something like

$$= x^6 + 4x^5 + 2x^4 - 4x^3 + x^2 - 1.$$

The notation $g(f(x))$ is read "g of f of x" or, perhaps more transparently, "g after f of x." Frequently the function g in $g(f(x))$ is referred to as the "second" function because it is not evaluated at any value until after f, the "first" function, has been evaluated for a given x-value. However, referring to this ordering may be troublesome for students since when reading from left to right the first function appears to be g. "Inside" function and "outside" function work when describing the symbolic form, but are not as meaningful from a graphical view.

In a traditional approach, students must recognize the potential limitations of the composition by identifying symbolically the domain and range of the first function and then determining whether or not that range is a subset of the domain of the second function. Often the process is presented in a diagram similar to that in Figure 1.

This model is very abstract and most likely not very meaningful to high school students. The TI-92 offers a visual way to explore composition of functions by tracing the path of a domain element through the functions being composed. The activity presented here provides students an opportunity to take advantage of the graphical representation of the process of composition in addition to a more traditional symbolic approach to the topic. The activity is intended to give the students an

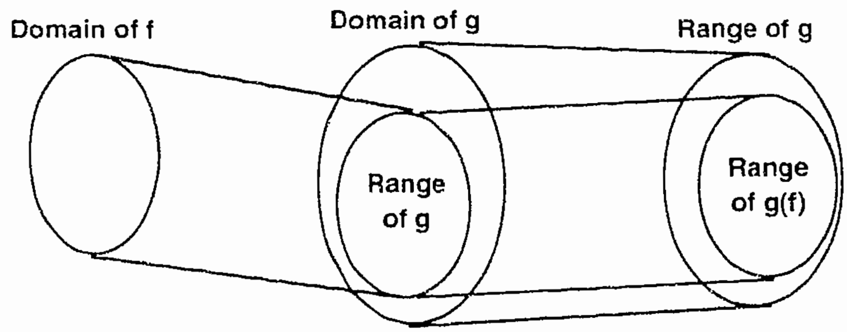


Figure 1. Diagram depicting the composed function, g(f) .

experience with composing functions so that the requirement of Range(f) $\subseteq$ Domain(g) is concrete for them. The use of the TI-92 to explore many examples may also help students to appreciate the distortion of the domain of the second function by the first.

The mathematical content of this activity is most likely no different from that of a traditionally taught high school advanced algebra course. However, the understandings that students can gain from this experience will be quite different. The meanings students associate with composition of functions will include taking the range of the first function and rotating it back to the x-axis to be re-evaluated.

This activity may require some introduction by the teacher, either at the overhead or on the board. Much would depend upon the students' familiarity with the function concept as well as with the TI-92. The notion of composing functions can be introduced through a realistic problem setting. An example of this regards the Tilapia, a versatile fish raised inexpensively for food in aquacultures. The population of Tilapia is a function of water quality as measured by the growth of algae. The quantity of algae in the pond is a function of temperature (among other things). Here I will focus on the more abstract setting of two continuous functions, devoid of context.

An Example

In this example, two functions, f and g, are defined in the home screen of the TI-92 calculator as is their composite (as displayed in Figure 2), with f going first. In this case, the Range(f)=Domain(g) so there is no concern about limitations placed on the composition. Two methods of exploring are discussed.

The two functions are graphed (see Figure 3a) using the displayed window settings (see Figure 3b). The purpose of this activity is to trace

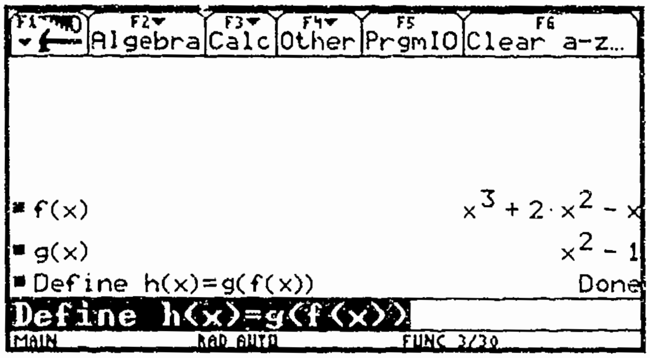


Figure 2. Definition of functions f , g , and h.

(a)

(b)

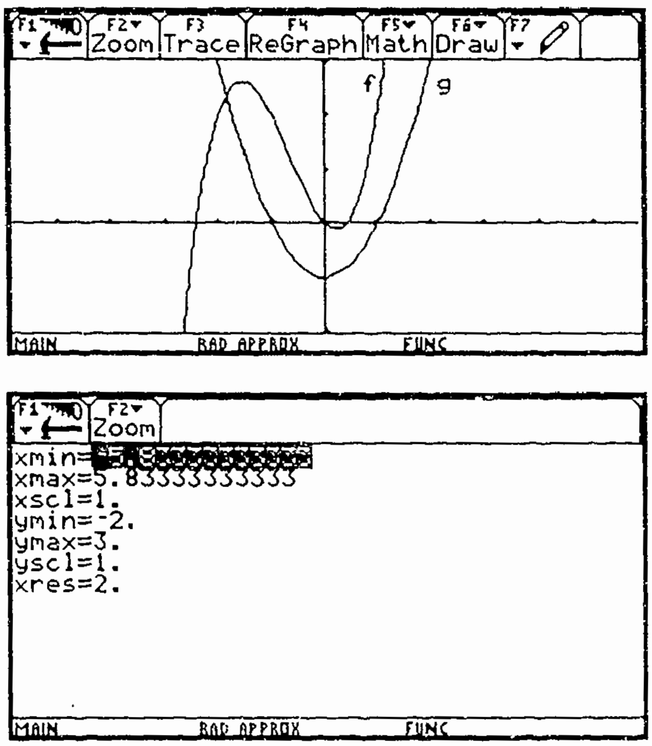


Figure 3. (a) Graph of functions f and g ; (b) Window settings for graph in (a).

on the graph through the composition for a specific x-value. After the first function is evaluated, it will be important to locate that function value on the x-axis in order to evaluate the second function there. A diagonal move of the cursor from the y-axis to the x-axis signifies the change from the range of the first function to the domain of the second.

Method 1

The purpose of this method of exploration is to trace an x-value from the domain of f (the first function) all the way through to the function value for the composition using the Line option (F7, 3) from the graph screen of the TI-92. The following path will be illustrated:

- Locate an arbitrary x-value from the domain of f, call it x_0, (on the x-axis start at the point $(x_0, 0)$. Note that approximations must be tolerated.).
- Draw a line from $(x_0, 0)$ to the point $(x_0, f(x_0))$ on the graph of f.
- Draw a line from $(x_0, f(x_0))$ to $(0, f(x_0))$, moving horizontally to the domain of f.
- Draw a line from $(0, f(x_0))$ on the y-axis to the point $(f(x_0), 0)$ on the x-axis (this will most likely require an approximation of the values):
- Draw a vertical line to the point $(f(x_0), g(f(x_0)))$ on the graph of g.
- Draw a horizontal line to meet the original vertical at the point $(x_0, g(f(x_0)))$.

To illustrate this method, I have arbitrarily chosen -0.686... from the domain of f to be the value of x_0, and will illustrate the method by showing how it applies to $g(f(-0.686...)$. Figure 4a shows the cursor at this initial point and the prompt "1st point?" This indicates that the Line function has been chosen from the F7 menu. In response to Enter, the calculator prompts for "2nd point." Note that this command remains activated until the user hits Escape or invokes another command. Figure 4b shows where the cursor is in a general composition diagram.

After responding to the calculator prompt, the cursor is moved to the second point of this line segment, the point (-0.686..., f(-0.686...)) as shown in Figure 5a.

Next, a line is drawn horizontally from the curve to the function value, f(-0.686...), on the y-axis as shown in Figure 6a. (This requires marking (-0.686..., f(-0.686...)) as the first point and (0, f(-0.686...)) as the second point.) This reinforces the notion that the input for the second function is attained from the range (y-values) of the first function (see Figure 6b).

Figure 7 shows the switch from the range of the first function to the domain of the second depicted by a line from (0, f(-0.686...)) to the function value, f(-0.686...), on the x-axis (the point (f(-0.686...),0), see Figure 7a).

Drawing a line vertically to g, we find the function value under g for this intermediate x-value ($f(x)$), which is shown in Figure 8a.

(a)
(b)

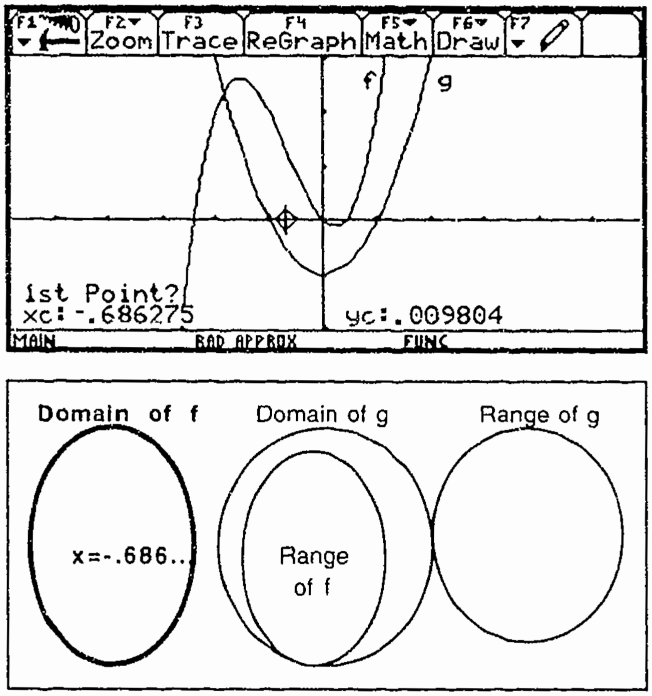


Figure 4. Starting from the domain of f with the domain element x = -0.686...

(a)
(b)

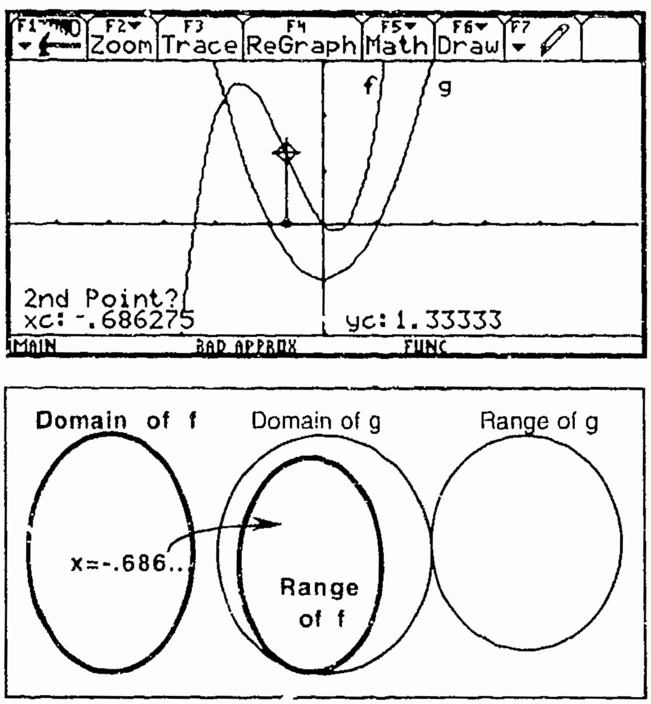


Figure 5. Moving from (-0.686..., 0) to (-0.686..., f(-0.686...)) where f(-0.686...) = 1.333....

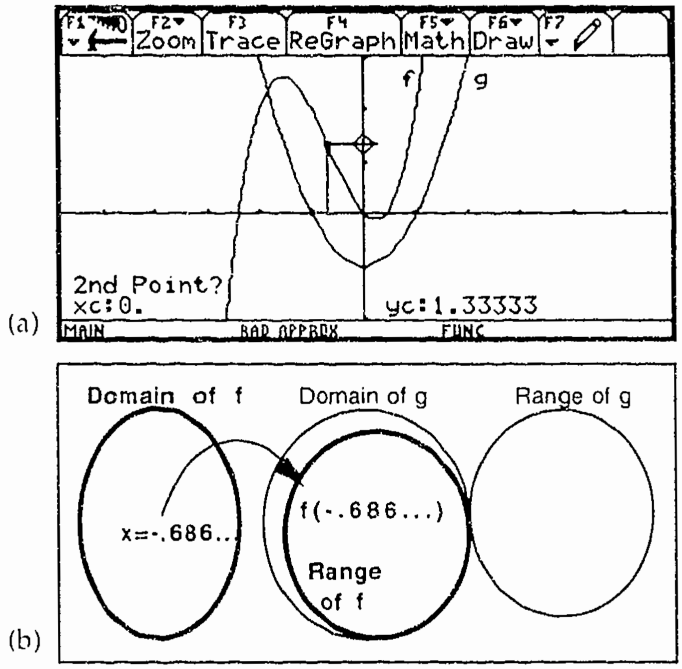


Figure 6. Locating the function value for f on the y-axis.

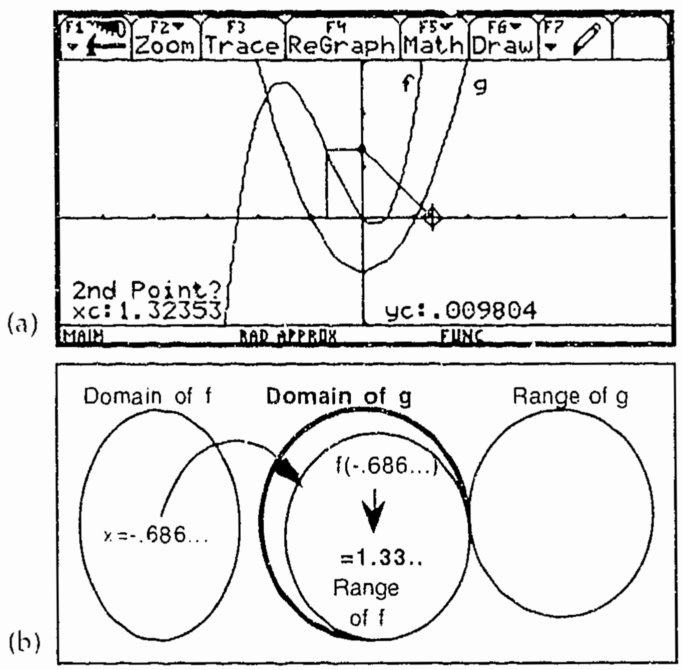
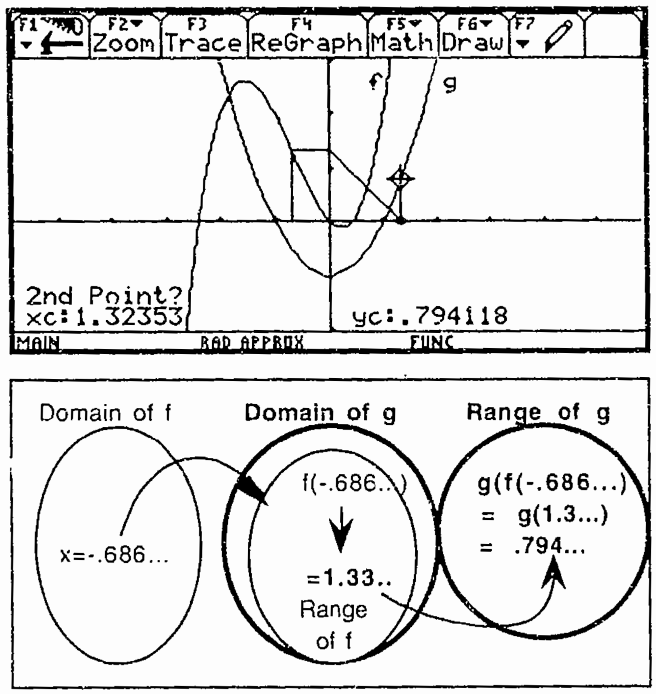


Figure 7. Graphically converting f(x) to a domain element for the function g, here f(-0.686...) is thought of as 1.333... .

(a)



(b)

Figure 8. Locating the function value g(f(x)), here g(f(-0.686...))=g(1.333...).

By drawing a horizontal line (left or right), we can see our line intersect the y-axis. This is the composite function value for our beginning x-value. The point of intersection of the original vertical (where $x_0 =$ -0.686...) and the last horizontal segment is a point on the curve of the composite function, shown in Figure 9.

There is no way on the TI-92 to save this point without it being identified as the intersection of the lines. Although there is an eraser function, it is very easy to erase more than just the auxiliary lines. One way for students to collect several points before graphing the composed function is to use a small piece of overhead transparency, cut to fit the TI-92 screen, and record the points with a felt pen or overhead pen (to preserve the screen). After recording each point on the transparency, the lines can be erased with the command F6, 1: ClrDraw. This way students can plot several points before graphing $h(x)$ to see how they did.

It may be helpful in this activity to have the x-axis and y-axis set at the same scale. The standard window settings on the TI-92 do not accomplish this. Note that setting the scale to square allows for the diagonal move from the y-axis back to the x-axis. The resulting diagonal line may help students to think of the intermediate step as a rotation of the y-axis down to the x-axis.

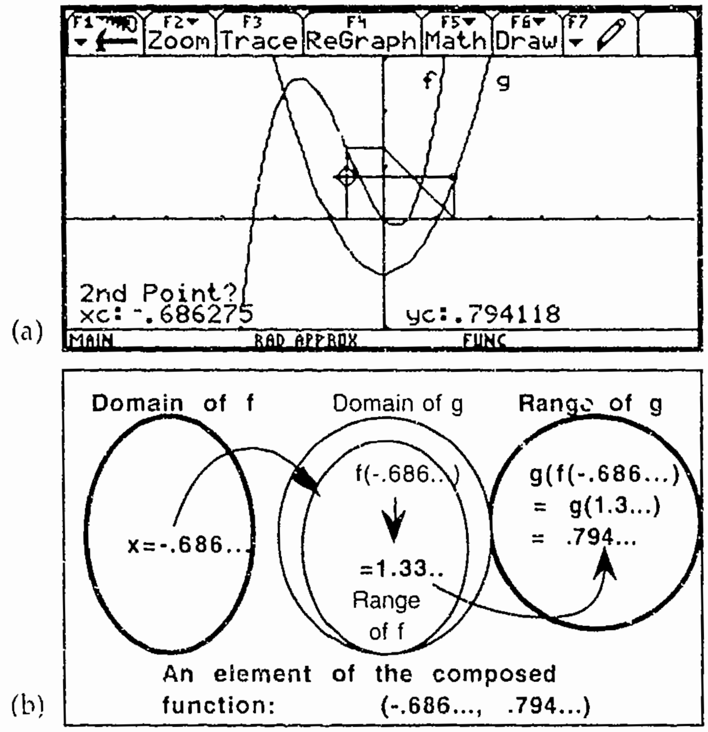


Figure 9. Associating the value of g(f(x)) with the input x.

Once the students are familiar with this idea, they can try another method of locating some points in the composite function. The purpose of this method is to get students to think about the effect of the first function as sort of folding or distorting the domain of the second function.

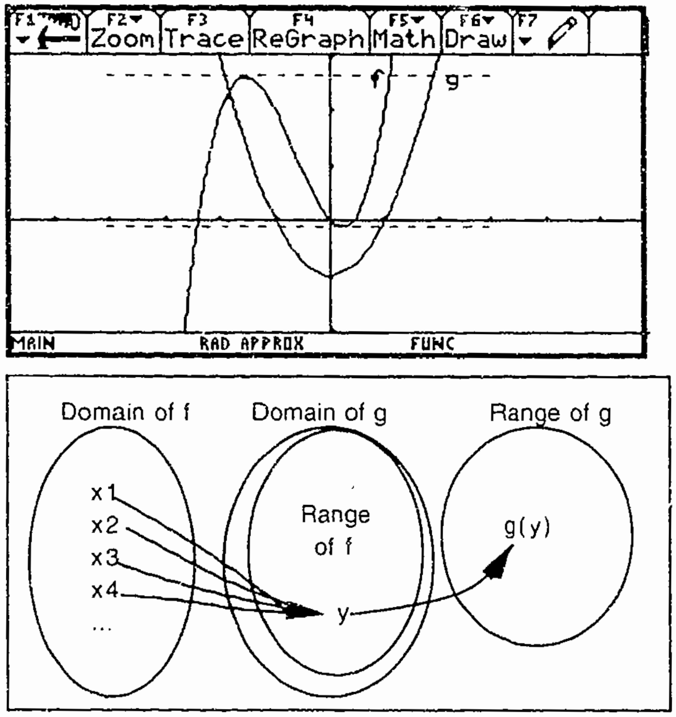
Method 2

From the same starting graph and window (Figure 3), we observe that if we were to think of projecting *f*, the first function, onto the *y*-axis, we can name the range of *f* (in this case, all real numbers). Another thing we notice is that some parts of the graph of *f* map to the same subset of the range of *f* (see Figure 10), that is, to the subset of the range for which *f* is not one-to-one.

This method entails starting with a value from the domain of *f* such as x_1 from Figure 10. We will locate $y=f(x_1)$ and place a horizontal line through the point $(x_1, f(x_1))$, thereby locating other domain elements which map to $y=f(x_1)$. We will follow this path:

- Locate an arbitrary *x*-value, call it x_0, from the subset of the domain of the function *f* for which *f* is not one-to-one.
- Move vertically to locate the point $(x_0, f(x_0))$.

(a)



(b)

Figure 10. (a) The portion of the range of function f which is not one-to-one. (b) A diagram of the mapping in the region.

- Draw a horizontal line through the point $(x_0, f(x_0))$ and notice where the line intersects the graph of f. We will see that each x which maps to $f(x_0)$ will map to the same $g(f(x_0))$.
- Locate the point $(f(x_0), 0)$ on the x-axis.
- Move vertically to the graph of g, namely, to the point $(f(x_0), g(f(x_0)))$.
- Record the value of $g(f(x_0))$. Move the cursor to a different intersection of the horizontal line with the graph of the function f.
- Place small circles at each of the points $(x, g(f(x_0)))$ for which $f(x) = f(x_0)$.

This method begins similarly to Method 1 without drawing a line on every move. Figure 11a shows the cursor on the x-axis at $x_0 \in$ Domain(f), where x_0=0.882... ; Figure 11b shows that the cursor has been moved vertically to the graph of function f.

Using F7, 5: Horizontal, a horizontal line is placed through the point $(x_0, f(x_0))$=(0.882..., 1.382...) and then the cursor is moved horizontally to the point (0, f(0.882...)) on the y-axis as shown in Figure 12.

(a)

(b)

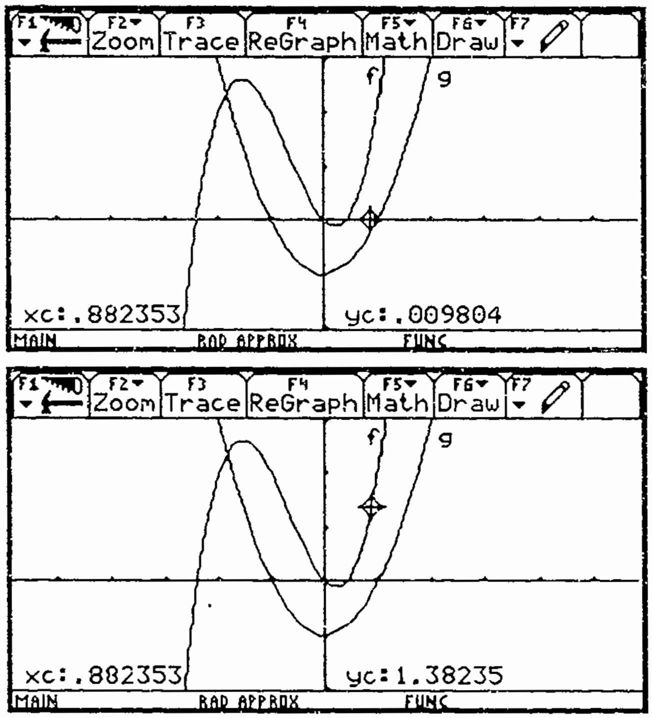


Figure 11. (a) The cursor at the beginning x-value, here x_0=0.882... . (b) The cursor moved vertically to f(x_0), the point (0.882..., f(0.882...)).

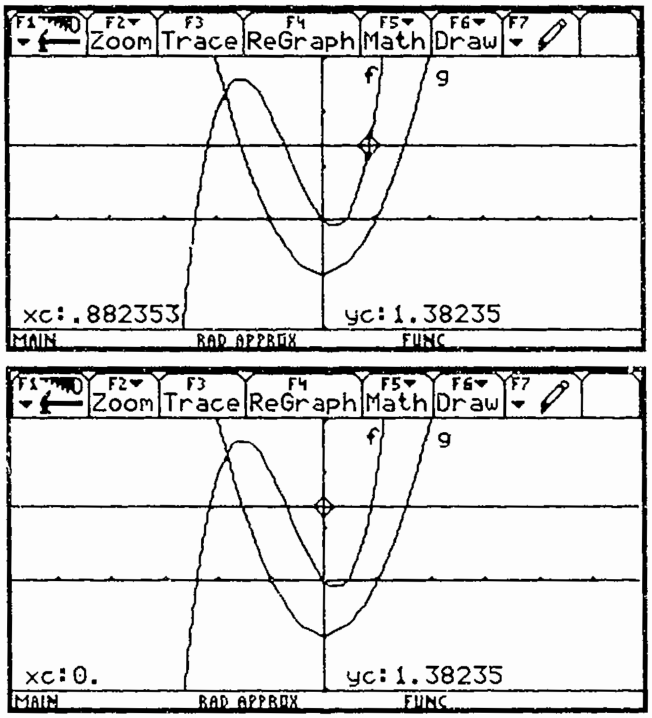
We move the cursor from the point (0, $f(x_0)$) to ($f(x_0)$, 0) on the x-axis (see Figure 13).

By moving the cursor straight up to the second function we locate the point ($f(x_0)$, $g(f(x_0))$). Figure 14 shows the result of this move and the final function value $g(f(0.882...))=0.843...$.

However, x = 0.882... was not the only x-value for which $f(x)=1.38...$. By moving the cursor to an intersection of the graph of f and the horizontal line, other such x-values are discovered. While maintaining that x-coordinate shown at the bottom of the graph screen, the cursor is moved vertically to where $y = g(f(0.882...))$ (see Figure 15).

A small circle is made to mark each of the composed points (x_i, $g(f(0.882...))$), (with $f(x_i)=f(0.882...)$) by using F7, 4: Circle. In order to do this, with the cursor at the point (x_i, $g(f(0.882...))$), hitting Enter marks the center of the circle; pushing the cursor out one click produces a circle of very small radius and hitting Enter again fixes the position of the circle. Figure 16 shows each of the three points marked with circles.

After returning to the Home Screen and graphing $h(x)$, the composite function appears on the screen and hits the predicted points (Figure 17).



(a)

(b)

Figure 12. (a) A horizontal line has been placed through the point (0.882..., f(0.882...)). (b) The cursor has been moved horizontally to the y-axis, depicting the function f domain element, f(0.882...) = 382...).

This method can be thought of as starting with an arbitrary x-value from the domain of the first function and identifying its function value. The second function is then evaluated at that function value. We may wonder if there are any other x-values which will be mapped to the same composed value. By looking at the intermediate value and working backwards, the answer can be found. In other words, which values of x make $f(x) = f(0.882...)$? Figure 18 shows how the CAS in the TI-92 can be used to easily answer this question.

The composition of these functions can also be studied symbolically on the TI-92 by substituting the rule for $f(x)$ in for the argument of g, or by evaluating g at $f(x)$, or by replacing the x in x^2-1 by the rule for $f(x)$. Each of these methods is shown on the screen in Figure 19. It can be easily observed that all are equivalent.

One of the important issues in composition of functions is the concern that the range (image) of the first function must be a subset of the domain of the second function. To remedy this situation, the domain of the first function can be restricted to that subset which maps to the domain of the second. For instance, in the first problem of the sample exercises shown in Figure 20, consider the composition $C(K(x))$.

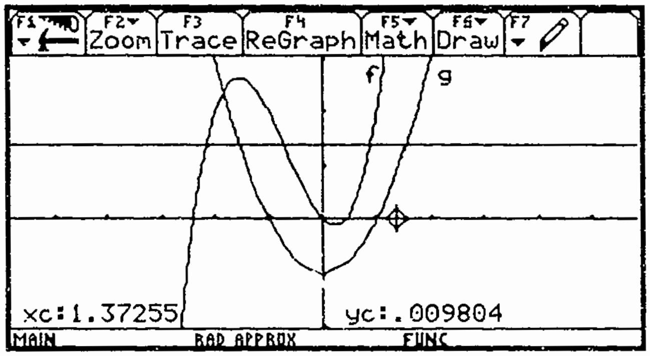


Figure 13. The cursor sits at the point (f(xo), 0).

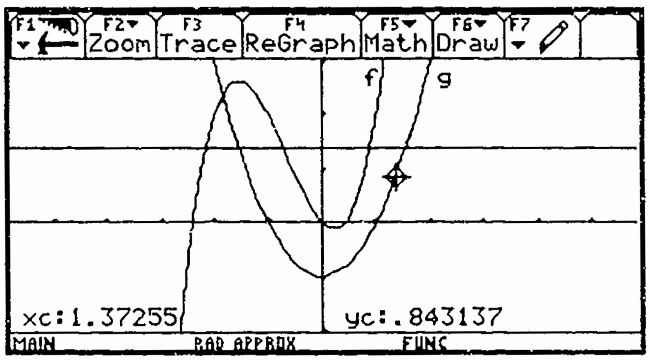


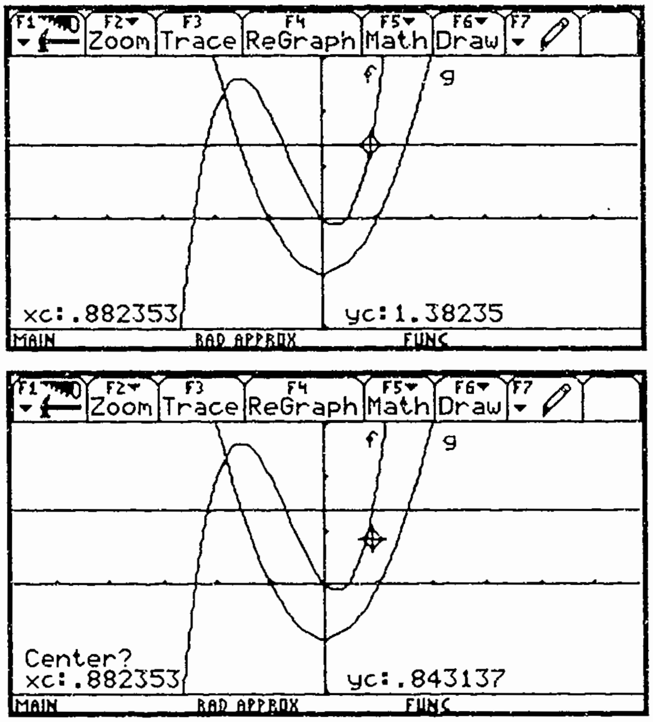
Figure 14. The cursor sits on the point (f((x_0), g(f(x_0)).

The image of K is [-5, ∞) and the domain of C is [0, ∞). Looking at graphs of the two functions helps make the issue more obvious. Figure 21a shows how these functions appear and (b) shows an unsuccessful graphical approach. In this case, the domain of the composition will be limited to values of x for which $K(x) \in [0, \infty)$:

$$\left(-\infty, -\frac{\sqrt{10}}{2}\right] \cup \left[\frac{\sqrt{10}}{2}, \infty\right).$$

Another case deserving of consideration is one in which the composed function actually extends the domain of the first function. Using the same two functions, C and K, the composition in the reverse order is $K \circ C(x)=2x-5$. The resulting function must be accompanied by the original restriction on the domain of C: $x \in [0, \infty)$.

Students should explore this concept with many different function combinations. They should be prompted to consider commutativity in composition and challenged to reason about why it does not always hold. Students can reason about why the composition of two linear



(a)

(b)

Figure 15. (a) An x-value which yields the desired composed value is identified. (b) The cursor is moved to the point (x1, g(f(0.882...)).

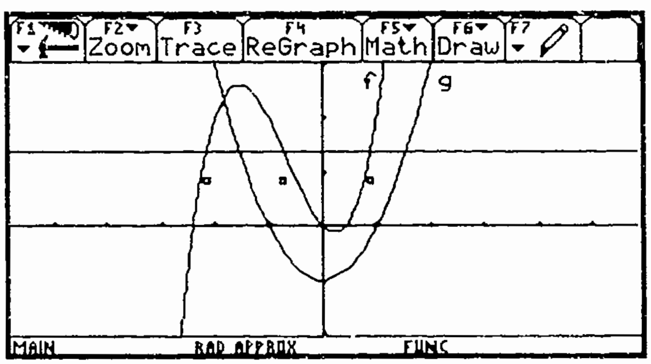


Figure 16. Circles marking the three points (x_i, g(f(0.882...)) where f(x_i)=f(0.882...).

functions is linear, but the composition of two other types of functions is not usually of the same type. These types of explorations can naturally motivate the fact that not all functions have inverses. Students can be helped to begin to think of functions as compositions of simpler functions. These types of activities can contribute to the development of their ability to reason symbolically.

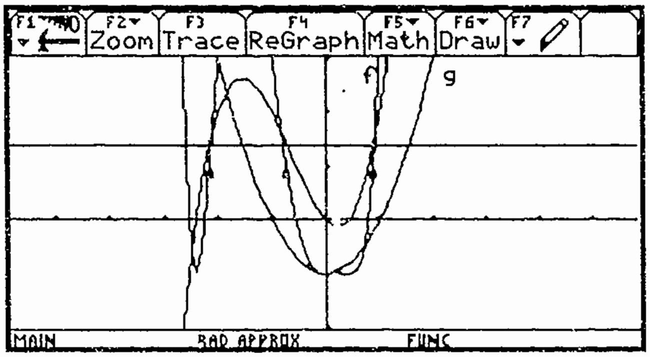


Figure 17. Graph of h = g(f) showing that h hits three points with the same function value.

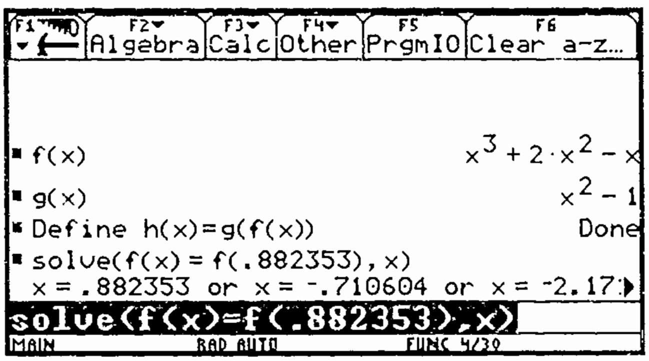


Figure 18. Solving for function values which map to the same composed function value.

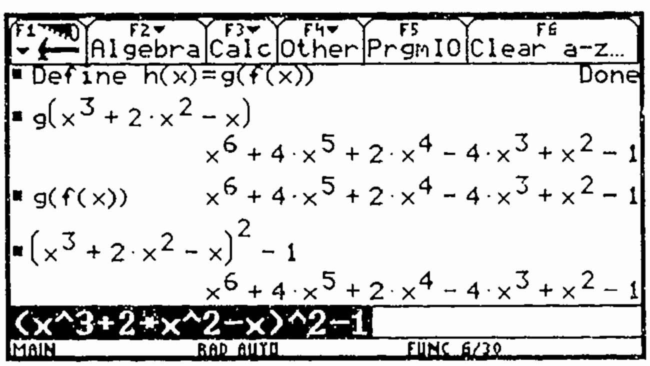
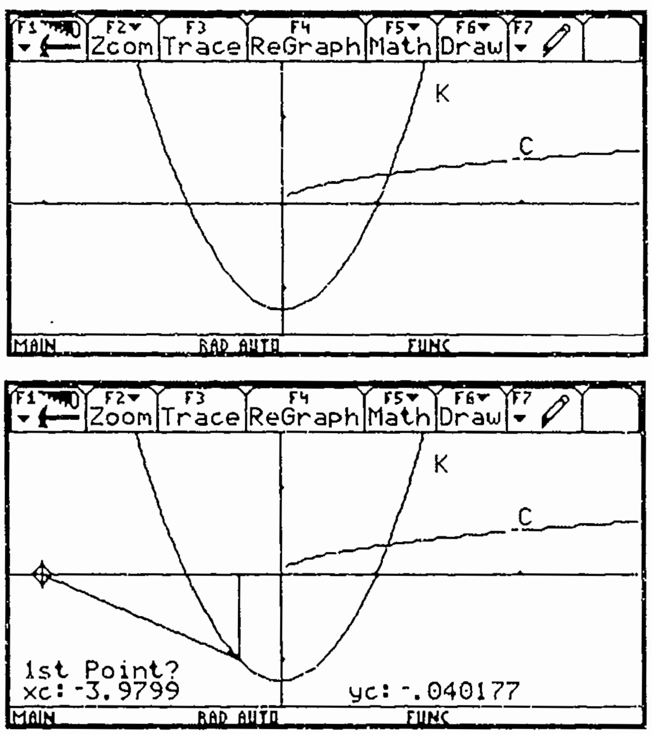


Figure 19. Using the CAS to show equivalent ways to compose functions.

1. = $2x^2 - 5$ $\qquad$ $C(x) = \sqrt{x}$
2. $w(x) = \sin(x)$ $\qquad$ $v(x) = \sin(x)$
3. $f1(x) = \dfrac{x}{x-3}$ $\qquad$ $f2(x) = \dfrac{3x}{x-1}$
4. Consider the function: $h(x) = \dfrac{1}{\sqrt{x^2 - 3}}$

   How might you think of this as a composition of functions? Explain.

Figure 20. Sample explorations.



(a)

(b)

Figure 21. An example of a case in which the domain of the composed function, C∘K, is not the same as the domain of K.

ABOUT THE AUTHOR

Linda Iseri is a former California high school mathematics teacher and a doctoral candidate in Mathematics Education at The Pennsylvania State University. Her address is 109 Rackley Building, University Park, PA 16802.

The Use of Technology in the Learning and Teaching of Mathematics

(continued from page 12)

Every student is to have access to a calculator appropriate to his or her level. Every classroom where mathematics is taught should have at least one computer for demonstrations, data acquisition, and other student use at all times. Every school mathematics program should provide additional computers and other types of technology for individual, small-group, and whole-class use. The involvement of teachers by school systems to develop a comprehensive plan for the ongoing acquisition, maintenance, and upgrading of computers and other emerging technology for use at all grade levels is imperative. As new technology develops, school systems must be ready to adapt to the changes and constantly upgrade the hardware, software, and curriculum to ensure that the mathematics program remains relevant and current.

All professional development programs for teachers of mathematics are to include opportunities for prospective and practicing teachers to learn mathematics in technology-rich environments and to study the use of current and emerging technologies. The preparation of teachers of mathematics requires the ability to design technology-integrated classroom and laboratory lessons that promote interaction among the students, technology, and the teacher. The selection, evaluation, and use of technology for a variety of activities such as simulation, the generation and analysis of data, problem solving, graphical analysis, and geometric constructions depends on the teacher. Therefore, the availability of ongoing in-service programs is necessary to help teachers take full advantage of the unique power of technology as a tool for mathematics classrooms.

The National Council of Teachers of Mathematics recommends the appropriate use of technology to enhance mathematics programs at all levels. Keeping pace with the advances in technology is a necessity for the entire mathematics community, particularly teachers who are responsible for designing day-to-day instructional experiences for students

National Council of Teachers of Mathematics (February, 1994)

SIGNIFIERS AND COUNTERPARTS: BUILDING A FRAMEWORK FOR ANALYZING STUDENTS' USE OF SYMBOLS

Margaret Kinzel
The Pennsylvania State University

That high school students have difficulty with algebra has been well documented (Kieran, 1992). Much has been done to identify and classify students' "errors" in algebra; that is, the ways in which students' use of algebraic symbols does not match "correct" procedures. This approach has served to articulate the differences between students' incomplete conceptions and more desirable conceptions, but it only addresses part of the problem. The other part involves finding ways to help students construct more appropriate understandings. Saying the same things louder and slower may result in more appropriate applications of memorized procedures, but this approach fails to address the difficulties encountered from the students' perspectives. It is pessimistic to believe that a major activity of algebra students is randomly writing meaningless marks on paper. A more generous (and reasonable) perspective is that students are involved in genuine efforts to make sense of an abstract notation system. From this perspective, the student's actions have reasons that at least attempt to mirror actions modeled by others, and that, to the student, make sense. Our task as teachers then becomes one of understanding the reasons that underlie students' actions. Only with that understanding can we create activities that we hypothesize will engage the student in examining and refining his or her own conceptions. To illustrate this approach, consider the fairly routine task of finding the equation of a line that passes through two given points. Take a minute to think through how you would approach this task, then read on.

One possible solution strategy involves using the x- and y-coordinates of the given points to find the slope, which is then substituted in to the $y=mx+b$ form of a linear equation. One point is chosen and its coordinates substituted for x and y to determine the value of b. Once the value of b is found and substituted into the equation, x and y reappear as variables and we have the "answer."

Now consider it from a potential student perspective. At times, the letters x and y stand for general coordinates of any point that satisfy the given conditions, while at others, x and y are replaced by specific values; the student is left to determine if x and y are variables (quantities that vary and are related by a linear rule in this case) or unknowns (letters that stand for specific numbers). The parameters m and b are waiting to have values substituted for them (unknowns), but when x and y have been "plugged in," b becomes the "variable" to be solved for. We shift from one interpretation of the symbols to another and back without explicitly discussing how or why (Usiskin, 1988). Fluent users of algebraic symbols often do not attend to these implicit shifts in meaning within a particular task. Small wonder, then, that students have difficulty following as we flip back and forth seemingly at random.

Mathematics educators have long pointed out the inherent duality in mathematical symbols (Gray and Tall, 1994; Kaput, 1987; Mason, 1987; Sfard, 1991; Skemp, 1976). In order to develop fluency with algebraic symbols, one must develop an awareness of what to attend to and when. As in the example above, attention often shifts between general and specific interpretations of the symbols within a single task. Those already fluent with symbols may barely attend to these shifts. In contrast, those struggling to follow a process may remain unaware that a shift has occurred and thus lose sight of the overall approach to the task. Tall's "proceptual thinking," for example, involves being able to see an expression as both a process and a product–the ability to interpret $2x$-3 as the process of multiplying a value by two and subtracting three as well as a representation of the result of that process. The expression is then seen as both a computation that can be carried out and a quantity that may be operated on. Proceptual thinking implies the awareness of both possibilities and the ability to choose and shift between them. Pimm (1995), building on Schmidt (1986), uses *signifier* and *counterpart* to capture the dual nature of mathematical symbols. These terms can be applied more specifically to algebraic symbols and provide the basis for a framework for analyzing students' use of symbols.

Signifiers and Counterparts in Algebra

As signifier, a symbol points to the object being considered; it names or calls the object into being. A signifier leads one's attention *away* from the symbol itself, towards the thing being named. Using symbols to label the quantities in a problem or to record the actions taken are examples of attending to the signifier aspect of symbols. Referring to the equation of a line task, the signifier aspect of an expression is evident in the formula for the slope. The expression is waiting for values to be substituted in, and the attention is drawn towards the potential result, not to the expression itself. If a symbol is interpreted *only* as a label or record, it is not seen as an object in its own right, and thus is not available as an input for manipulation or for reasoning. The notation is simply a record of an operation that has been, or could be, carried out, given values for the letters within the expression.

The counterpart aspect of a symbol draws one's attention *towards* itself, as something to operate on or with. Counterparts can be manipulated as if they are the objects in question; that is, the symbol "holds" the meaning, freeing the student from having to focus on the meaning, and thus allows the student's focus to shift to the operations. For example, the Arabic numerals "hold" the abstract numeric quantities, allowing manipulation with the symbols to reflect potential actions on those quantities. This shift to a focus *on the symbol* allows for efficient manipulation; "setting aside" a tight connection to the referent to allow for fluid operations. The flexibility to see an expression as a representation of the result of an operation shifts attention to the counterpart aspect of the expression as a symbol. If the expression for the slope had been substituted into the $y=mx+b$ form of the equation in the above example, this could be evidence of a shift in attention to the counterpart aspect of the expression. This is perhaps a trivial example of such a shift since the signifier aspect is dominant: the expression is seen as an entity that may be substituted into another expression (counterpart), but it is still a process waiting to be carried out (signifier).

Traditional algebra instruction has focused on the rules governing operations *on* symbols, emphasizing the counterpart aspect. Students often come away with sets of memorized rules with little or no understanding of how or why the rules are applied in various situations. The ability to see both aspects of algebraic symbols and to shift between them as required by a task is a crucial component of symbolic reasoning.

In addition to shifting between the signifier and counterpart aspects of a symbol, there can be shifts in the meaning or referent for the symbol. In the preceding equation of a line task, the referents for x and y shift within the solution, from general labels for the coordinates of *any* point on the line to specific labels for the coordinates of a particular point. This shift from general to specific feels trivial to those fluent in the interpretation and manipulation of algebraic symbols, however this shift may not be obvious to novice algebra students. It is possible for students to follow the steps in the procedure without being aware of the implicit shifts of focus between meanings and aspects of the symbols. The task of managing (or even following) these shifts of attention can seem overwhelming.

The framework I propose involves assuming that students have reasonable motivations guiding their operations with algebraic symbols. Under this assumption, focusing on that to which the student is attending provides insight into these motivations. The language of signifiers and counterparts provides a structure for organizing observations of students' work.

Signifiers and Counterparts in Action: Exploring Students' Work

To investigate the usefulness of the signifier/counterpart language in discussing students' use of algebraic symbols, task-based interviews were conducted (and videotaped) with twelve undergraduate college students enrolled in the first of three courses focused on the teaching and learning of secondary mathematics. Each of these students had completed at least three semesters of calculus along with an introduction to proofs course. Participants were asked to work on two to four tasks (one at a time) and to share their thinking as they worked. The tasks were selected for their potential to provide opportunities for the students to struggle with the symbols; that is, the tasks did not lend themselves to the application of memorized procedures. A discussion of one task and the work of two students on that task is presented here.

Treasure Hunt Task

A treasure is located at a point along a straight road with towns A, B, C, and D on it in that order. A map gives the following instructions for locating the treasure:

i. Start at town A and go 1/2 of the way to C.
ii. Then go 1/3 of the way to D.
iii. Then go 1/4 of the way to B, and dig for the treasure.

If AB=6 miles, BC=8 miles, and the treasure is buried midway between A and D, find the distance from C to D.

Charosh, M. (1965). *Mathematical Challenges*, Washington, DC: NCTM.

Before proceeding to read the remainder of this chapter, the reader would do well to spend time investigating ways to solve the Treasure Hunt Task.

The Treasure Hunt Task has proven to be a rich context for observing students' use of symbols in this study. The mathematics involved is well within the range of secondary mathematics education majors. However the complexity of the symbolization required makes it a non-trivial exercise. Part of this complexity comes from the multiple steps in the problem; the expression resulting from each step is used as an input for the next step, compounding the expression and increasing the level of abstraction away from the concrete components of the problem. In addition, as one works on the task, it is necessary to interpret expressions in terms of both distances and locations. As a distance, an expression can be added or subtracted to produce expressions for other distances. Yet it is also necessary to locate the result of traveling the distance on the line in order to determine subsequent distances.

In this paragraph and the one that follows it, I will describe a typical solution to the Treasure Hunt Task, a composite of the solutions I have seen or used in work on this problem. Initially, symbols are used as signifiers to represent the situation posed in the task. A line segment is drawn to signify the road, with dots or marks to indicate the towns and the given distances labeled with numerals. Already, a shift in focus can be identified in the use of the line segment in that it both represents the stretch of road (signifier) and is also available for manipulation (counterpart), as it is divided into sections between towns. In addition, the labels for distances AB and BC are almost trivially treated as counterparts to produce a combined distance for AC, which is then operated on to determine the result of the first instruction (Start at town A and go halfway to C.). At this point, the signifier and counterpart aspects of the symbols are closely linked; it is not necessary to step too far away from the referent in order to carry out these initial manipulations.

Beginning with the second instruction, however, the symbols take on more complexity. Part (ii) asks for one-third the distance to town D, but part of that distance (CD) is unknown. An expression similar to $\frac{1}{3}(7+x)$ is easily generated, but can prove difficult to interpret. Seen only as a record of the process that *would* produce the actual distance traveled if CD were known, the expression is not viewed as an entity that can be used to continue work on the task. A shift from this signifier aspect to the counterpart aspect (a quantifiable distance) is needed in order to combine with the 7 miles traveled in part (i) and produce an expression for the total distance traveled. The resulting expression represents the total distance traveled, but also names the location, on an imagined coordinatized number line, reached by traveling that distance. Both interpretations need to be available in order to proceed with part (iii). Adding this third step makes the task complex enough to be a non-routine exercise in symbolization. To solve the problem, the sum of distances traveled in parts (i) to (iii) is equated with the distance traveled straight to the treasure (half the distance from A to D), again requiring a flexible interpretation of expressions both as distances traveled and locations reached.

In this task students are asked to make multiple shifts between aspects of the symbols, in the interpretations of those symbols, and to manage these shifts through several levels of abstraction (further away from concrete values). The following section summarizes the work of two students on this task. Although each of these students' work is intended to be accurately represented in the descriptions that follow, the reader should not assume that the names or the genders of these two students are revealed in those descriptions.

Pat

Pat's work on the Treasure Hunt Task illustrates effective management of the symbols within the context. Throughout her interview, Pat is able to articulate what her symbols represent and frequently stops to interpret her expressions in terms of the context. Having first labeled distances on the line segment representing the road, she soon changes to thinking in terms of coordinates to facilitate her symbolization. With the exception of one instance, Pat consistently shifts focus between distance and location to appropriately generate a new expression.

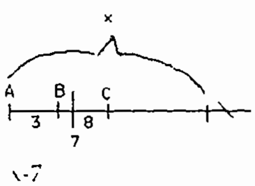
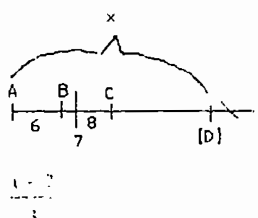
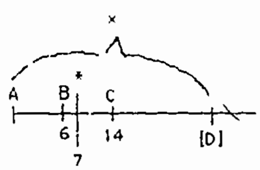
Pat's work	Pat's words	*Interpretation*
	Let's see, so I'm at 7, definitely. I want to go one-third of the way to D, so D min---or x, why don't I just name it D, then I'll know. x minus 7 [Pat writes x-7.] will give me the distance between where I am—[Pat places * above 7 on drawing]	*7 is signifier (distance traveled in part (i)) and counterpart (used to find next distance)* *The line segment is both signifier (label for context) and counterpart (symbol being manipulated to reflect further actions).*
	---and D [Pat points to vertical mark which is not yet labeled as D]. So I want one-third of that [Pat puts x-7 over 3]. That'll give me this. That'll give me where I should be at that point [after part (ii)] with relation to this.	*x-7 is a label for the remaining distance to D; it is seen as a quantity that can be divided by 3 to find one-third of that distance.* *Pat interprets expressions as distances and locations; the flaw seems to center on her assuming that $\frac{x-7}{3}$ is the treasure-hunter's location after part (ii).*
	And the quarter of the way back to B. B is at 6. Hmm. Oh—Hmm. Okay, I kind of have an idea. It should all this, these three things [steps in the problem] should, if I can fit them into some sort of equation that says, tells me where to go and like, puts it where it should be, then it should be equal to halfway, okay. It should be halfway between A and D, and the Ds will probably cancel out, it'll look real nice, and—All right, I'm not sure if I like what I have done now.	*As a result of this reasoning, Pat shifts to thinking in terms of coordinates on the line; changes labels on line to reflect coordinates* *Pat coordinates interpretations of distances and locations to hypothesize a solution strategy*

Table 1. Pat's work on the Treasure Hunt Task.

Pat's approach to this task is to symbolize the distances (both known and unknown) and to manipulate those symbolic representations in a manner that is consistent with the nature of the context. Distances can be interpreted as both the actual distance traveled as well as a means of locating the result of traveling that distance. Initially, Pat shifts easily and implicitly between these interpretations, operating with 6 and 8 to produce 7 as both the distance traveled in part (i) and the location reached as a result of traveling that distance. At this point, the symbols are fairly transparent; their connection to the context is very close to the surface. As the expressions become more opaque (their interpretation in terms of the context is more abstract), Pat shifts more consciously between aspects of the symbols. Pat returns to the signifier aspect of the expressions to reattach contextual meaning in order to support her manipulations of the expressions. In doing this, Pat also shifts between thinking in terms of distances and locations. This shift is accompanied by adjusting her diagram to reflect the coordinate interpretation. That Pat is aware of these shifts and uses them to support her reasoning is clear in the next excerpt: Pat focuses on the signifier aspect to assure herself of the appropriateness of her equation, then consciously releases that focus to "just simplify."

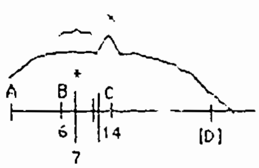
Pat's work	Pat's words	*Interpretation*
 6 24 3 [illegible]	Okay, if it's less than 25—All right. So now what? If I add 6 to this, then it'll be in terms of the entire line, I think. It should be	*Expressions are counterparts: Pat does not attend to their referents in the process of simplifying.* *Pat changes from D to d: possibly indicative of a shift to the counterpart.* *Pat shifts attention to the signifier aspect: she reasons from context for the next step*
$4d - 94 = \frac{d}{2}$	4d minus 94 is equal to d over 2 [Pat writes 4d-94=d/2], which is half of the distance between there. Between my things. Okay, so I just have to simplify	*Pat consciously shifts to counterpart aspect, she knows she can "just simplify."*
$4d - 94 = \frac{d}{2}$ 8d-188=d 7d=188 d=26 14 [illegible]	[Inaudible. Pat is solving the equation] D equals 2, 26. D is located at 26, the answer is [Pat writes 14 under 26; 12 underneath] 12. The distance from C to D is 12.	*Expressions are counterparts as Pat solves the equation, she returns to the signifier aspect to interpret the result.*

Table 2. Pat's work on the Treasure Hunt Task.

This analysis of Pat's work illustrates the use of the signifier/counterpart language as a window into students' symbolic reasoning. By attending to where Pat's attention seemed to be focused, we see how shifting between aspects of symbols can support reasoning processes. Pat's conscious awareness of the shift to counterpart to facilitate manipulation suggests the benefits that might be gained through making these shifts a more explicit focus of classroom discourse. A final thought from Pat's work is more subtle. As she shifts to simplifying the expression in the first section of the above table, she changes the symbol "D" to "d." While this may be nothing more than notational convenience, the fact that it accompanies a shift in her thinking raises a question. Does "D" embody the signifier aspect (a label for the town) while "d" more closely implies the counterpart aspect (a component of an expression)? It is unclear from the available data if this shift was significant for Pat, or if she even was aware of the change in notation.

Janis

In contrast, Janis finds it difficult to maintain the connection between her symbols and their referents. She initially has trouble interpreting her expression for part (ii) $\frac{1}{3}(7+x)$ as an entity that would allow her to progress to part (iii). Rewriting this expression as $\frac{7+x}{3}$ seems to facilitate shifting to the counterpart aspect, and Janis is able to proceed, although she is unclear as to what this fraction represents. In moving ahead, she first approximates the location this would indicate on the line, but then struggles with an interpretation and a potential next step.

Janis' approach, like Pat's, begins with symbolizing the distances within the problem. However, Janis encounters difficulty much sooner than Pat, and her difficulty can be discussed in terms of signifiers and counterparts. Janis easily symbolizes the distance traveled in part (ii) as $\frac{1}{3}(7+x)$ but continues to focus on the signifier aspect of this expression. As evidenced in the last comment in the above excerpt, Janis is uncomfortable with interpreting this (or its equivalent fraction representation) as the product that would result from the process of computing the distance traveled. Rewriting her expression in fraction form seemed to support a tentative step towards interpreting $\frac{7+x}{3}$ as representing a value. It is possible that the consolidated fraction form is more easily interpreted as a value (as opposed to the more process-oriented $\frac{1}{3}(7+x)$).

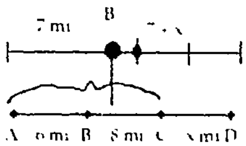
Janis' work	Janis' words	*Interpretation*
 a) AC=6+8=14 mi $\frac{14}{2}=7$ mi	If just approximating, I guess you would be somewhere about here. [Janis divides segment from heavy dot to D into thirds and puts small dot on the first third]	*Line segment is counterpart; Janis is able to perform actions on it as if it were a stretch of road. She interprets $\frac{7+x}{3}$ as the treasure-hunter's location after part (ii); it is more appropriately interpreted as the distance traveled in part (ii).*
b) 1/3(7+x) = $\frac{7+x}{3}=7$ $\frac{7+x}{3}=7+x$	My problem is in deciding how I can get this [(7+x)/3] to equal where I am between C and D. For some reason I think this [Janis writes =7 next to (7+x)/3] should equal 7. Oh, this would be---[Janis writes 7+x beside (7+x)/3=] No. [Janis scribbles out.] I was thinking it would equal 7+x because this is the total distance between B prime and D. But we're not really setting up equation to equal somethin--	*Janis seems confused over what represents; she wants it to equal something in order to proc...l: she appears to want to shift to the counterpart aspect but she is unable to generate an appropriate equation.*
c) $\frac{7+x}{3}\cdot\frac{1}{4}=\frac{7+x}{12}$	I want to say for part c, all you do is multiply one-fourth times all this $\frac{7+x}{3}$, but I don't think that's right either. [Interviewer: Talk about that.] I don't think it would work because it's looking at the distance between B prime to—It's comparing the distance from B prime to D and not the distance between this new point [second dot] and C. And I don't think that's the same thing, but I don't know. [Janis writes for part (iii): [(7+x)/3] 1/4=(7+x)/12] But see, then how do you solve for x? Because you don't know what's over here. That's where I'm confused. That's why up here I think it should equal something, so we have a value of how far you went	*Janis shifts to the counterpart aspect but in so doing loses connection to referent.* *Janis attempts to interpret in terms of the line* *Janis carries out manipulations without clear connection to the referent.* *Janis is still uneasy with the counterpart aspect of $\frac{7+x}{3}$*

Table 3. Janis' work on the Treasure Hunt Task.

With her tentative shift to the fraction form, Janis is able to pursue symbolic manipulations. This shift in manipulation is not accompanied by a corresponding shift in interpretation, and Janis quickly loses contact with the referents for her symbols. Sensing that further manipulation is needed, Janis pushes ahead with part (iii) but is clearly at a loss for a suitable interpretation for the result of her operations. In an effort to simplify the task, Janis draws a smaller segment of her original diagram and focuses on this.

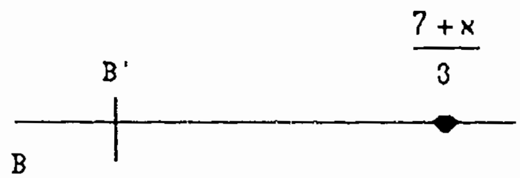


Figure 1. Janis' diagram of smaller segment.

This seems to help Janis focus on the remaining steps of the task, but in so doing, essentially prevents her from finding an appropriate solution. This limited focus severs the remaining connection to the context; ignoring the whole line detaches Janis' expressions from their appropriate referents. Janis is unaware of this and she continues to work on symbolizing the remaining steps, interpreting her expressions in terms of her limited diagram. Janis has confidence in her manipulative skills and appears more comfortable as she uses her smaller diagram to support further manipulations of her symbols and eventually arrives at a final expression, $\frac{11+5x}{12}$. Clearly pleased with this reasonable looking result, Janis boxes this expression and declares it the answer. However, she has no clear interpretation for this expression in terms of the context and seems unaware that she has not answered the question posed by the task (Find the distance from C to D.).

In Janis' case, we see the result of not maintaining clear connections to the referents for the symbols. Janis remained unaware of the implicit shifts in her own work as well as the missed opportunities where shifting focus may have supported her reasoning about the task. Her manipulations lead her to a result so far removed from the context that she was unable to trace back through the symbols for an interpretation. As with Pat, Janis' work indicates that making shifts in the focus of attention an explicit part of the problem-solving process could support students' development of symbolic reasoning.

Discussion

The Treasure Hunt Task interviews provide a context for observing students' use of symbols in a problem-solving context. Throughout their work we see the interaction between the dual aspects of symbols: a symbol is introduced as a signifier, attention is shifted to the symbol as counterpart to different degrees within the task, and attention must be returned to the signifier aspect within the problem-solving process and at the conclusion of the task. In addition, the task requires flexible interpretations of the expressions generated within the task: shifting between distances and locations as referents for the expressions. Focusing on the dual nature of mathematical symbols through the signifier/counterpart language provides a means for attending to what the student attends to, and thus serves the investigation of the student's symbolic reasoning.

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ABOUT THE AUTHOR

Margaret Kinzel is a doctoral candidate in Mathematics Education at The Pennsylvania State University. Her address is 118 Rackley Building, University Park, PA.

Calculators and the Education of Youth
A Position Statement of the National Council of Teachers of Mathematics

Calculators are widely used at home and in the workplace. Increased use of calculators in school will ensure that students' experiences in mathematics will match the realities of everyday life, develop their reasoning skills, and promote the understanding and application of mathematics. The National Council of Teachers of Mathematics therefore recommends the integration of the calculator into the school mathematics program at all grade levels in classwork, homework, and evaluation.

Instruction with calculators will extend the understanding of mathematics and will allow all students access to rich, problem-solving experiences. This instruction must develop students' ability to know how and when to use a calculator. Skill in estimation and the ability to decide if the solution to a problem is reasonable and essential adjuncts to the effective use of ihe calculator.

Evaluation must be in alignment with normal, everyday use of calculators in the classroom. Testing instruments that measure students' understanding of mathematics and its applications must include calculator use. As the availability of calculators increases and the technology improves, testing instruments and evaluation practices must be continually upgraded to reflect these changes.

The National Council of Teachers of Mathematics recommends that all students use calculators to—

explore and experiment with mathematical ideas such as patterns, numerical and algebraic properties, and functions;

develop and reinforce skills such as estimation, computation, graphing, and analyzing data;

focus on problem-solving processes rather than the computations associated with problems;

perform the tedious computations that often develop when working with real data in problem situations;

gain access to mathematical ideas and experiences that go beyond those levels limited by traditional paper-and-pencil computation.

(continued on page 74)

EXPLORING CONTINUED FRACTIONS: A TECHNOLOGICAL APPROACH

Thomas A. Evitts
South Western High School

Continued fractions are "multiple-decked fractions" (Olds, 1963, p. 7) that have forms similar to those shown in Figure 1. When every fraction in the expression has a numerator of 1, the fraction is called a simple continued fraction (See Figure 1(b).). In addition, continued fractions may be infinite in nature–an endless stack of numerators and denominators, often showing a repetitive sequence (See Figure 1(c).).

(a) **finite** continued fraction

$$1+\cfrac{3}{2+\cfrac{3}{7}}$$

(b) finite **simple** continued fraction

$$2+\cfrac{1}{4+\cfrac{1}{4}}$$

(c) **infinite** simple continued fraction

$$2+\cfrac{1}{4+\cfrac{1}{4+\cfrac{1}{4+\cdots}}}$$

Figure 1. Examples of finite continued fraction, finite simple continued fraction, and infinite simple continued fraction.

Continued fractions have a rich history in the development of mathematical ideas. The use of continued fractions to solve indeterminate[1] equations can be traced to sixth century India (National Council of Teachers of Mathematics (NCTM), 1989; Olds, 1963). It is also possible that rational approximations for irrational numbers that appear in early Greek writings may be derived from continued fractions (NCTM, 1989, pp. 100, 267). Olds (1963) cites Rafael Bombelli (16th century) as the originator of the modern theory of these fractions. Other mathematicians prominent in the investigation of continued fractions are Cataldi, Wallis, Huygens, Euler, Lambert, Lagrange, and Stieltjes (NCTM, 1989; Olds, 1963).

In addition to their appeal as classroom exercises in rational number computation, continued fractions provide opportunities for students to explore sequences of real numbers and make connections across several topics in secondary mathematics. These topics (including the real number line, convergence of sequences, interpretation of graphs of functions (See Bevis & Boal, 1982.), inductive reasoning, and iterative processes) constitute a mathematical core often encountered by students and teachers willing to extend their exploration beyond simple algebraic manipulation. That calculations related to continued fractions can be performed on a computer or calculator affords deeper and richer possibilities for the inclusion of this typical "enrichment topic" in the curriculum.

Students need to be or become familiar with the following concepts and skills to engage successfully in the explorations suggested in this chapter:

- notation and meaning of proper and improper fractions;
- notation and meaning of a reciprocal, including, preferably, the use of -1 as an exponent;
- understanding of the meaning of infinite, as used in the sense of an infinite continued fraction and in repeating decimals and transcendental numbers;
- ability to perform an iterative process and record appropriate results as they occur; and
- knowledge of solving quadratic equations and finding irrational roots.

The proposed activity may actually lend itself to introducing these ideas.

Each of these is likely to be introduced, encountered, and reinforced in a meaningful way as part of a larger mathematical context.

Using the Texas Instruments TI-92 Calculator[2]

In particular, the TI-92 seems well-suited to handle a number of explorations focused on the renaming of continued fractions as real numbers and vice versa. While students may choose to tackle the expression in Figure 1(a) "by hand," the TI-92 can be used, as illustrated in Figure 2. The student may enter the expression using nested parentheses: 1 + (3/(2 + (3/7))); when the "pretty print" mode is engaged, the history area of the screen displays the full representation of the continued fraction. In addition, students might opt for a more step-by-step approach, using an exponent of -1 for the reciprocal. This is illustrated in Figure 3. Clearly, as in the case of the continued fraction in Figure 1(b), when all numerators are equal to one, these repeated steps illustrate an iterative process.

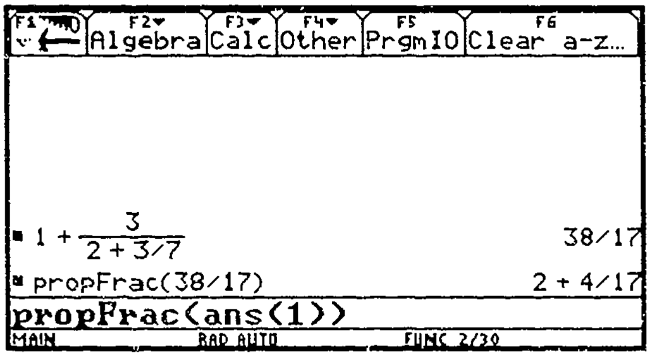


Figure 2. TI-92 "pretty print" display after the following expression is entered: 1 + (3/(2 + (3/7))). The propFrac operator then transforms the improper fraction into a mixed number.

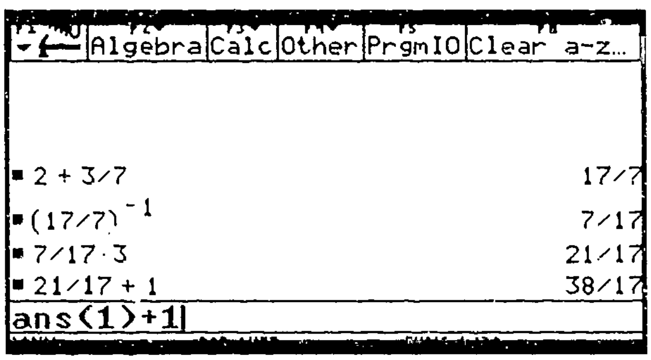
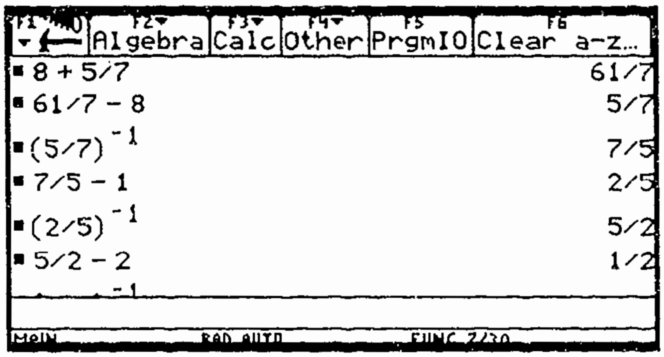


Figure 3. The use of a step-by-step approach on the TI-92 to evaluate the expression, 1 + (3/(2 + (3/7))).

An attempt to reverse this process and write a mixed fraction as a continued fraction requires the students' attention to the concepts employed to evaluate a continued fraction. A student may attempt to "undo" the steps described above. To produce the continued fraction values for the mixed number $8\frac{5}{7}$, the sequence of TI-92 steps shown in Figure 4 may be used. Note the use of an iterative process: subtract the largest whole number that allows for the difference to remain above zero, take the reciprocal of the remaining fraction, repeat until the resulting fraction has a numerator of 1. The students will have to record, as they work, the significant numbers necessary to make the continued fraction, namely the three subtracted whole numbers and the remaining fraction. Taking the reciprocal one last time results in 2, the final denominator. Continued fractions are often written in terms of these whole numbers; thus, $8\frac{5}{7}$, could be denoted as [8,1,2,2] (or [8;1,2,2]).



The answer is:

$$8+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2}}}$$

Figure 4. Illustration of the sequence of TI-92 steps that may be used to produce continued fraction values for $8\frac{5}{7}$.

Mathematical Connections

This notation and its iterative derivation are noteworthy for several reasons. First, the appearance of the same sequence in the Euclidean algorithm, useful for finding greatest common divisors, presents an opportunity for mathematical inquiry into the apparent connection (See Figure 5.) between continued fractions and the Euclidean algorithm. Kimberling (1983) elaborates on this and suggests ways for students to utilize computer programs centering on the algorithm. Through the use of a program that generates the continued fraction notation, students can examine patterns and properties of continued fractions (p. 511). While not included in this article, a programming approach using the TI-92 may be an appropriate and appealing direction to consider.

$$61 = \mathbf{8} \times 7 + 5$$
$$7 = \mathbf{1} \times 5 + 2$$
$$5 = \mathbf{2} \times 2 + 1$$
$$2 = \mathbf{2} \times 1 + 0$$

Figure 5. The Euclidean Algorithm carried out with 61 and 7.

Second, exploration of the Euclidean algorithm invites student conjecturing and theorizing. The possibility exists for students and teachers to raise and discuss a number of important mathematical questions. Do all rational numbers have a finite representation? Are finite simple continued representations unique? How would one evaluate or approximate irrational numbers using continued fractions?

One of the dilemmas that students will face in encountering the infinite continued fraction (e.g., Figure 1(c)) is its lack of closure; the strategy of the finite continued fraction, starting with the "bottom" denominator and working up through the expression, fails here. An approach which generates a list of successive convergent values seems appropriate. [1, 1, 1, 1, . . .] denotes a continued fraction useful for illustrating the iterative derivation of convergents on the TI-92 (See Figure 6(a).). The calculator not only prints the entered expression in the continued fraction form but allows continued expansion through the user's insertion of an additional "1+1/(" in front of and ")" in back of each expression, thereby reinforcing the concepts of a continued fraction and the iterative process. Figure 6(b) illustrates the power of the TI-92 in this endeavor.

Students can explore the potential of the TI-92 to utilize more efficient iterative processes and produce successive approximations for the value of the fraction. Two possible approaches are illustrated in Figure 7. Both require students to think of variable as more than an unknown and, in the second case (Figure 7(b)), a different notion (and notation) for function are encountered. Students with more robust images of both variable and function should have little trouble devising and using these strategies.

The TI-92's capability to express numerical values in both fraction and decimal forms is particularly useful. The fractions obtained are interesting and instructive because of the appearance of the Fibonacci numbers, but they do not offer the student a sense of convergence of the successive approximations. Changing the mode or using the approximation key is useful for illustrating that this sequence of values seems to converge (See Figure 8.).

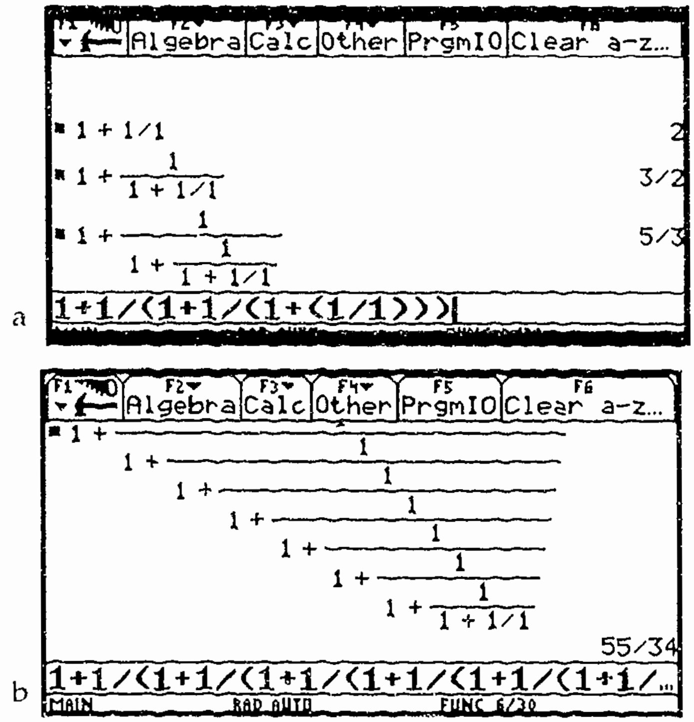
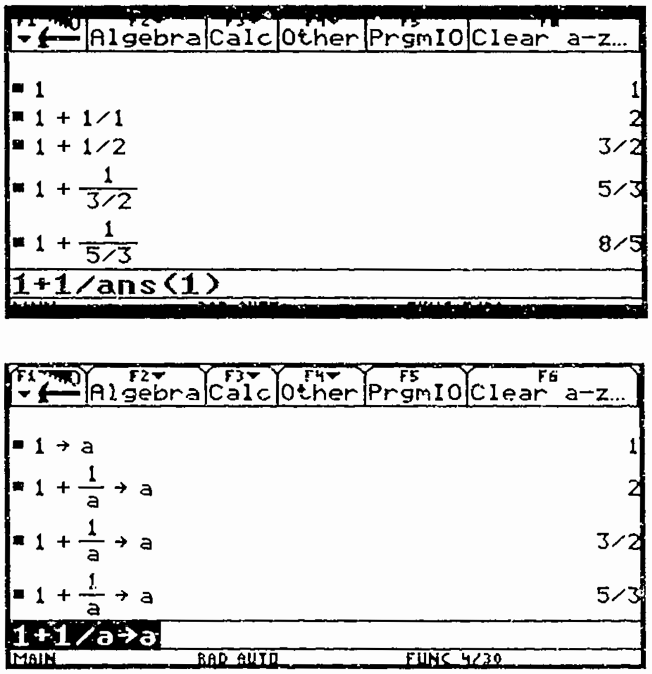


Figure 6. Illustration of the continued expansion of an expression through the user's insertion of "1 + 1/(" in front of and ")" in back of each expression.

Emphasizing Algebra

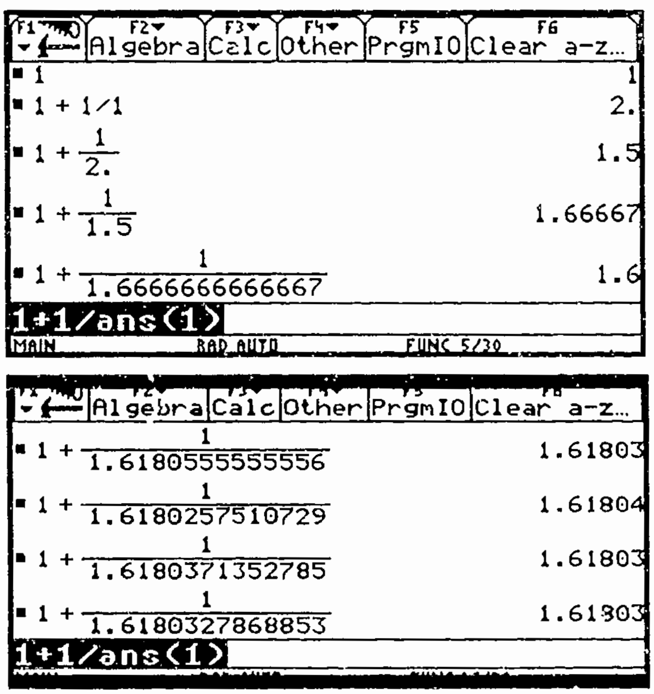
Creating simplified and algebraically useful forms, namely, $x = 1 + \frac{1}{x}$ and the equivalent $x^2 - x - 1 = 0$, from these iterative processes can be accomplished rather easily, depending on the capabilities of the student. These equations and the embedded relationship of a number and its reciprocal having a difference of one contain appropriate algebraic content and concepts for students. The TI-92 is useful in solving the above equations and can generate both the exact answer and a familiar decimal approximation. Figure 9 illustrates these solutions, obtained by first placing the calculator in EXACT mode and then in APPROXIMATE mode. The rich connections of the Fibonacci numbers to the Golden Ratio may also not be apparent or familiar to students. The teacher is likely to play a significant role at this stage in the activity, through a variety of means–for example, questioning individuals and small groups about the relationships they see or bringing the class together to discuss findings and entertain conjectures.



a

b

Figure 7. Two approaches to producing successive approximations for the value of the continued fraction, [1, 1, 1, 1, ...].



a

b

Figure 8. Producing successive decimal approximations for the value of the continued fraction, [1, 1, 1, 1, ...].

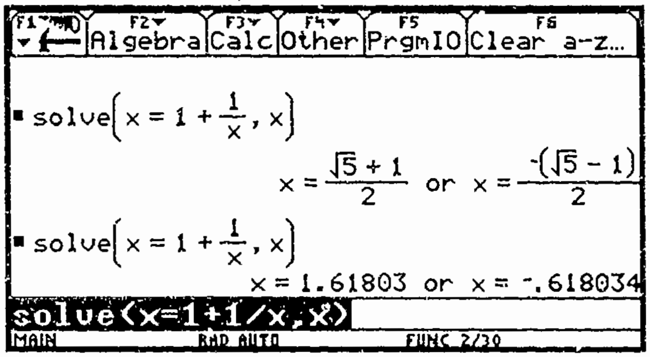


Figure 9. Exact and approximate solutions to the equation $x = 1 + \frac{1}{x}$ obtained by first placing the TI-92 calculator in EXACT mode and then in APPROXIMATE mode.

Exploring complex forms allows students to extend their understanding of iterative processes. For example, the continued fraction [2; 4, 4, 4, . . .] produces a series of easily found convergents on the TI-92 using symbolic iteration, as shown in Figure 10(a). The iterative function approach to produce the same sequence of convergents is slightly more sophisticated. One method is illustrated in figure 10(b). The motivation for a programming alternative to this task is present again.

This process leads to the question of finding an equation from which a radical form for this number can be obtained. The equation will be a quadratic equation with integral coefficients (Stevenson, 1992, p. 134). In a manner similar to that for $x = 1 + \frac{1}{x}$, the equation, $x - 2 = \frac{1}{4 + (x - 2)}$ can be rewritten to produce $x^2 - 5 = 0$; thus $x = \sqrt{5}$. A host of conjectures and questions regarding the continued fraction form for $\sqrt{5}$ arise and make for interesting explorations. Figure 11 illustrates how substitution creates an equation from which algebraic roots can be found.

In turn, the search for the continued fraction form of any square root offers yet another perspective on iteration and additional patterns to explore (See Stevenson, 1992; Masunaga & Findall, 1993). Students can easily begin with an equation of the form $x^2 = n$ and begin by finding the greatest integer $\sqrt{n}$ as the first numeral in the continued fraction expansion. The process employs several algebraic skills and continues in a

b

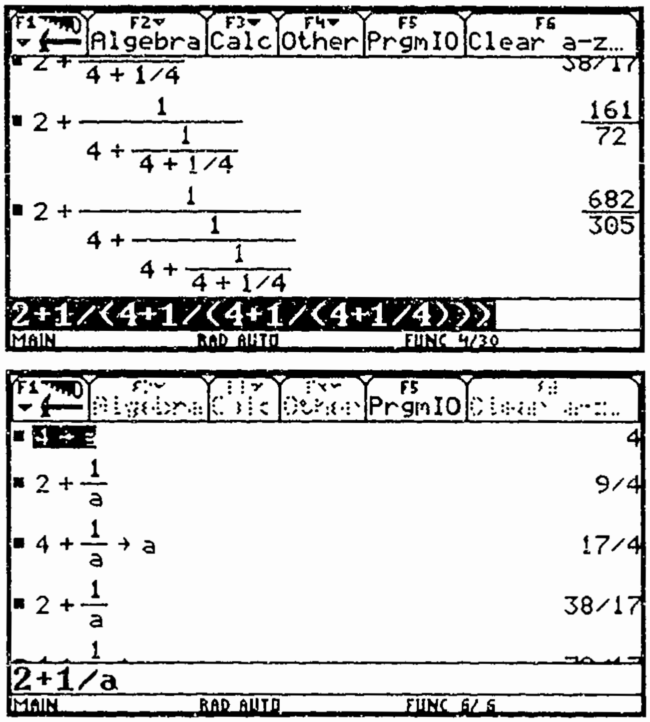


Figure 10. Sequence of convergents produced (a) by symbolic iteration and (b) by the iterative function approach.

$$x = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \dots}}} \dashrightarrow x = 1 + \frac{1}{x}$$

$$x = 2 + \cfrac{1}{4 + \cfrac{1}{4 + \cfrac{1}{4 + \dots}}} \rightarrow x - 2 = + \frac{1}{4 + (x - 2)}$$

Figure 11. Illustration of how substitution in a continued fraction can create an equation from which algebraic roots can be found.

a manner shown in Figure 12. The final result, obtained after repeating the algebraic steps until a recurring pattern is evident, yields the expansion [3; 1, 1, 1, 1, 6, 1, 1, 1, 1, 6, . . .] for $\sqrt{13}$. This periodic form can be written as $[3; \overline{1,1,1,1,6}]$. Olds (1963) generalizes about all square roots of non-perfect square integers when he remarks: "the continued fraction expansion of any quadratic irrational is periodic after a certain stage" (p. 56). This is known as Lagrange's theorem, first proved in 1770 (Olds, 1963, pp 56, 110-111).

$$x = 3 + (\sqrt{13} - 3)$$

$$x = 3 + \cfrac{1}{\cfrac{1}{(\sqrt{13} - 3)}}$$

$$x = 3 + \cfrac{1}{\cfrac{1}{(\sqrt{13} - 3)} \quad \cfrac{(\sqrt{13} + 3)}{(\sqrt{13} + 3)}}$$

$$x = 3 + \cfrac{1}{\cfrac{(\sqrt{13} + 3)}{4}}$$

$$x = 3 + \cfrac{1}{\cfrac{(4 + \sqrt{13} - 1)}{4}}$$

$$x = 3 + \cfrac{1}{1 + \cfrac{\sqrt{13} - 1}{4}}$$

$$x = 3 + \cfrac{1}{1 + \cfrac{1}{\cfrac{4}{\sqrt{13} - 1}}}$$

$$\ldots$$

Figure 12. Illustration of finding a continued fraction form for $\sqrt{13}$

Considering that other irrational numbers must therefore have non-terminating, non-periodic expansions raises some interesting parallels to the decimal expansion of fractions and introduces the idea of limit. Students may enjoy expanding, through successive approximations, several continued fractions of historical interest. As shown in Figure 13, Euler's expansion (1737) of $e - 1$, for example, is [1; 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, . . .] ; Lambert's expansion (1770) of π is [3; 7, 15, 1, 292, 1, 1, 1, 2, 1, 3, 1, 14, 2, 1, 1, 2, 2, 2, 2, 1, 84, 2,. . .] (Olds, 1963, pp. 135-136). These should intrigue students and may test the limits of their calculators! Gullberg (1997) states that "continued fractions converge more rapidly than power series expansions" (p. 144) and are thus very useful in approximating irrational numbers. Both Olds (1963) and NCTM (1989) provide additional examples and historical connections.

$$e - 1 = 1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \ldots}}}}}$$

$$\pi = 3 + \cfrac{1}{7 + \cfrac{1}{15 + \cfrac{1}{1 + \cfrac{1}{292 + \ldots}}}}$$

Figure 13. (a) Euler's expansion (1737) of $e - 1$ and (b) Lambert's expansion (1770) of π.

Pedagogical Implications

Access to the TI-92 eliminates much paper and pencil calculation and allows deeper exploration of the iterative process without errors. As illustrated above, the emphasis on iteration and the variety of ways to engage the calculator makes this a rich classroom exercise. The infinite fraction [1; 1, 1, 1, . . .] and its golden value is situated nicely at a variety of levels in the study of algebra, and technology allows students to encounter this number frequently in their mathematical work. Ar ably, this number should be as visible as π and e in the secondary curriculum.

The algebraic idea of reciprocal plays a critical role, especially in the $\frac{1}{a}$ or a^{-1} form. Both familiar symbolic notations have meaning in this exploration. The use of the calculator in efficiently performing repeated, identical calculations suggest both the obvious computational benefit and the necessity for students to carefully reason and explore their strategy as results appear.

As alluded to earlier, these explorations have definite mathematical goals undergirding them. Students are not completely led to "what they should do" and can approach these tasks in several ways. Essential to success are student interaction and teacher facilitation. Thoughtful, open-ended questions, brief interviews, and careful probing can be useful in assessing student (both individual and class) progress and in guiding without telling. Because of this topic's many connections, the teacher must be knowledgeable not only about the calculator but about the mathematics involved. It is likely that the teacher and students will be co-learners in this exploration; this presents a powerful image to students, one not usually encountered in secondary mathematics classrooms, about learning and listening. It places a heavy responsibility on the teacher to be genuine in his or her inquiry stance while also knowledgeable enough about the mathematics encountered to be credible to students. This is no doubt one of the most important aspects of teaching with technology.

It is likely that, without the TI-92, this topic would be reserved for an enrichment activity or rainy day distraction for upper level students. The TI-92 turns the seemingly novel idea of continued fractions into an exploration of some fundamental algebraic principles. Preparing teachers and encouraging students to cultivate broader and richer understandings of mathematics are essential if activity such as this is to yield its fullest results–serving as a springboard for further and deeper mathematical adventures.

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FOOTNOTES

[1] An indeterminate equation is one that has an unlimited number of solutions. For example $x + 5y = 10$ is indeterminate.

[2] One of the tools embedded in the TI-92 is a symbolic manipulation program. Other symbolic manipulation (like Derive, Mathematica, or Maple) could serve the same purpose.

ABOUT THE AUTHOR

Thomas A. Evitts is a high school mathematics teacher at South Western High School, 200 Bowman Road, Hanover, PA 17331.

THE ISOSCELES TRIANGLE: MAKING CONNECTIONS WITH THE TI-92

Karen A. Flanagan
The Pennsylvania State University

Ken Kerr
Glenbrook South High School

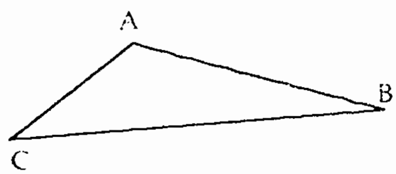
One of the recurring themes in the National Council of Teachers of Mathematics (NCTM) *Standards* (1989) is the idea of mathematical connections:

> Students who are able to apply and translate among different representations of the same problem situation or of the same mathematical concept will have at once a powerful, flexible set of tools for solving problems and a deeper appreciation of the consistency and beauty of mathematics. (p. 146)

Geometry from Multiple Perspectives (NCTM, 1991) suggests that these connections can be made in geometry through the blending of transformational, coordinate and synthetic approaches.

The concept of function is one of the most important concepts for students to understand, and much of this development now takes place in the context of algebra courses. Geometry can be one more place where this understanding can be developed. Goldenberg (1995) talks about a curriculum that teaches students to "think about theorems as functions" (p. 205). A theorem does not refer to a single static case, rather the theorem deals with a whole class of objects. Technology (such as the dynamic geometry tool[1]) is now available to help students think about theorems as statements referring to whole classes of objects. The Texas Instruments TI-92 calculator links a dynamic geometry system, a computer algebra system, a graphing utility and a table in one machine. This allows students to move easily between applications. Dynamic geometry makes it possible for students to generate classes of objects, collect data and examine what changes and what remains invariant. Data collection activities can allow students to create models and relate them to graphs. Using the curve-fitting capabilities of the calculator, students can obtain a symbolic form for the correspondence. These symbols can then be interpreted within the context of the original geometric object. With the TI-92's multiple linked representations it is quite easy to capitalize on those connections between geometry and algebra. An example of this using isosceles triangles follows.

Students can begin by exploring different relationships between angles and sides in a triangle using the measurement tools present in the TI-92. After examining several triangles they can characterize the relationship between the location of the longest side and largest angle. This can be done by first measuring the three sides and the three angles, and then dragging the various vertices to generate multiple sets of these measures with a goal of generating a conjecture about the longest side and largest angle. Relationships for the shortest side and smallest angle should also be examined. Students may discover relationships such as "the longest side of a triangle is opposite the largest angle, and the shortest side is opposite the smallest angle." Figure 1 shows one instance of this conjecture.



Largest: $\overline{CB}$ is opposite vertex of $\angle CAB$

Medium: $\overline{AB}$ is opposite vertex of $\angle ACB$

Smallest: $\overline{AC}$ is opposite vertex of $\angle ABC$

Figure 1. Illustration of the "longest (shortest) side / largest (smallest) angle" conjecture.

One can then ask students to determine what relationships occur between two of the sides. Students should be able to list the three possibilities: AC < AB or AC > AB or AC = AB. They can then use the calculator measurement utility to generate data from which they can conjecture relationships between the sides and the opposite angles. In the case of the isosceles triangle, data collection may result in the development of a symbolic rule for the measure of a base angle as a function of the measure of the vertex angle, or the inverse of that function rule which will output the measure of the vertex angle if the measure of the base angle is the input.

Students will begin by constructing an isosceles triangle. This drawing will not remain static. Therefore, to guarantee that if a vertex is dragged the triangle remains isosceles, either of the following methods may be used.

- Create a segment, AB, (Figure 2, left) and let the endpoints become two of the vertices of the triangle. Construct the perpendicular bisector of the segment, and the third vertex, C, can be placed anywhere along the perpendicular bisector.

- Another possible construction (Figure 2, right) involves creating a circle. Let the center of the circle, D, be one vertex and two points on the circle, E and F, be the other two vertices. The radii of the circle which contain the points E and F will become the sides of the triangle.

With the TI-92 you can hide the perpendicular bisector and circle so that all you see is the triangle.

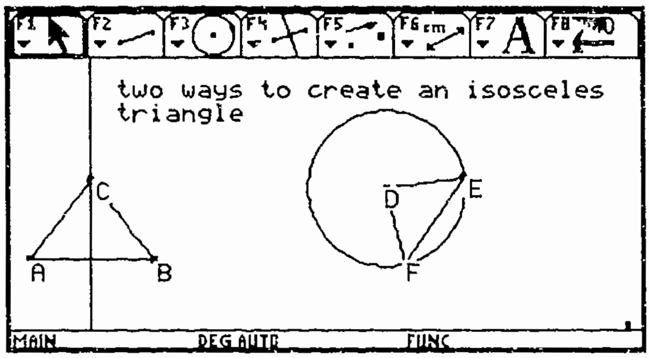


Figure 2. Two ways to create an isosceles triangle.

One of the greatest advantages of the TI-92 is the ability to bring diagrams to life. Once the isosceles triangle is constructed (See Figure 3.), the vertex can be dragged, changing the lengths of the sides of the triangle and the measures of the angles. Using the "Collect data" option in the TI-92 Geometry Application, data can be collected about the measures of the angles as the figure is altered through dragging (See Figure 4.). As students look at these measurements they can be asked to describe a relationship between the measure of the vertex angle and the measure of the base angle. Students can continue to alter the figure through dragging. As the figure changes the measurements change, and students can test their conjectures at each new position of the vertex.

These different sets of measurements can be stored in a table. The split screen option on the TI-92 allows one to view the geometrical object and view the collected data in a table at the same time. In this example the measures of the base angles were placed in the first and second columns, c1 and c2, and the measure of the vertex angle was placed in the third column, c3.

The data in the table can then be displayed in a scatterplot. Several questions about the process of creating a data display should arise naturally. When setting the viewing window, what should the maximum and minimum values be for the x- and y-values? Will the measure of the

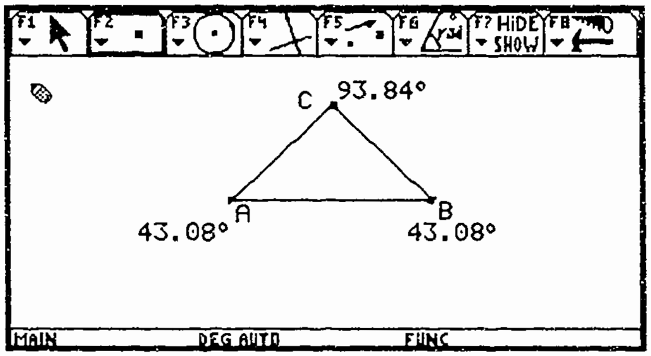


Figure 3. An isosceles triangle and its angle measures.

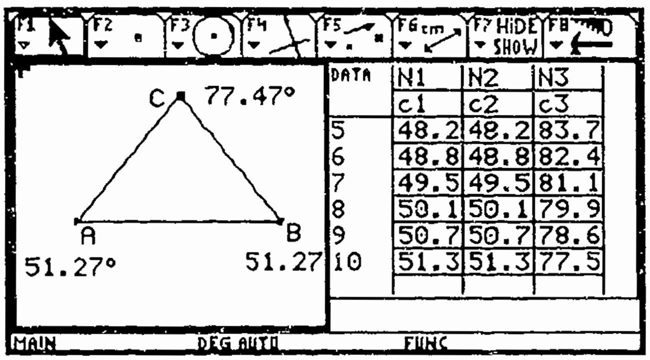


Figure 4. The split screen mode on the TI-92 shows the geometrical figure and the table of data values gathered through dynamic manipulation of that figure.

base angle ever exceed 90? Why? These questions can be used to foster discussion about some of the properties of an isosceles triangle. Students can also relate back to the ideas in algebra of choosing an appropriate domain and range. By using the fact that the sum of the measures of the angles of a triangle is 180, students may realize that the measure of the base angle should be between 0 and 90 and the measure of the vertex angle should be between 0 and 180. For the graph in Figure 5 of column 3 versus column 1, limits (xmin = -5, xmax = 100, ymin = -5, and ymax = 200) were chosen so that the axes are visible.

After viewing the data a decision can be made about what would be an appropriate function to fit to the data. The dat· appear to be linear. Is there anything from the context of the problem, the geometrical figure, that would indicate that the relationship might be linear? Again this can fuel some interesting discussions.

The graph of the regression equation appears on the screen (See

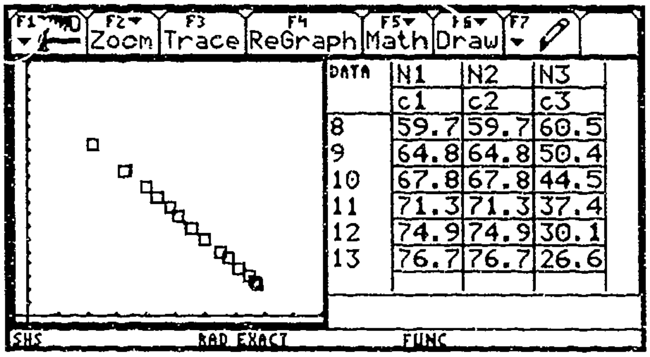


Figure 5. Split screen view of scatter plot and table.

Figure 6.) and appears to be a "good fit." We can now discuss the symbols that are attached to the graph (See Figure 7.). The vertex angle was recorded in x and the base angle in y; the equation is $y = -2x + 180$.

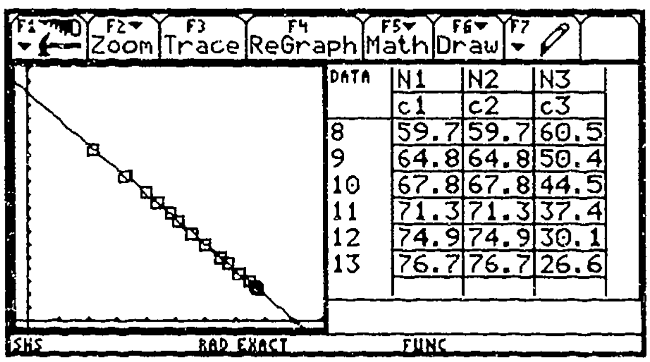


Figure 6. The graph of the regression equation relating base angle to vertex angle and the scatterplot.

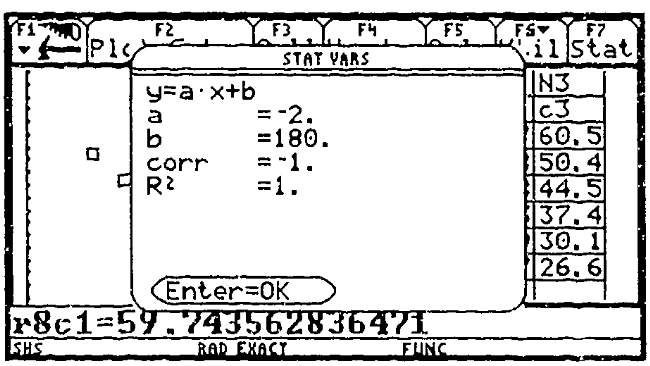


Figure 7. The regression equation relating base angle to vertex angle.

Discussion items could be: What does this tell us? How can we relate the symbols back to the context?

The domain and range issues for the fitted function can now be discussed. The line contains points which are not in the first quadrant. Is this legitimate for the isosceles triangle? Students may go back to the geometric figure and drag it to see if they can find an example of a base angle larger than 90° or the vertex angle larger than 180°. Students should try to explain the limits of the domain and range in terms of the changing shape of the triangle when it is dragged.

If students are allowed to decide what to choose as the dependent and independent variables, then there are three possible sets of input-output pairs from which to choose. One is base angle as a function of base angle, the second is base angle as a function of vertex angle, and the third is vertex angle as a function of base angle. A line which is fit to the base angle vs. base angle data plot will have rule $y = x$. It is important that students understand what is meant by these symbols. Students should realize this means the two angles of an isosceles triangle have the same measure.

The second possible combination of angles is one base angle and the vertex angle. If the base angle is graphed on the horizontal axis then the line fit to the data plot will be $y = 180 - 2x$. This function also must be interpreted with respect to the triangle involved. The subtraction of $2x$ suggests that there are two quantities that have the same value. To which two things are we referring? Since y was assigned to the measure of the vertex angle then $2x$ must represent the sum of the measures of the other angles. Since one of those base angles has measure x the remaining base angle must also have measure x.

An extension could be: What would happen if the columns were switched? For instance, if the vertex angle were placed in x instead of y what would the equation be? This is the third possibility: the measure of the vertex angle as a function of the measure of the base angle. Students should be challenged to decide for themselves whether the last two functions (i.e., the measure of the vertex angle as a function of the measure of the base angle and the measure of the base angle as a function of the measure of the vertex angle) are equivalent. This offers a nice opportunity to emphasize the role of input and output values in each of the function rules. The students could also be asked to reason about the meaning of the slopes in each case.

Whenever a data collection and curve fit has been done, the resulting mathematical model should be corroborated by some other means if possible. This can be accomplished by returning to our conjecture and the relationships that developed. The question can be raised, "What if none of the sides of the triangle are congruent?" What is true about the angles? How does this relate to the regression equations that were found in the isosceles triangle situation?

Another interesting property of the isosceles triangle is related to the altitude. Construct the altitude from the vertex angle of an isosceles triangle and label the point where it intersects the base. Measure the base and one of the segments that is formed on the base. The ratio of the measures is 2:1 (or 1:2 if the measurements were taken in the opposite order). What does this tell us about the altitude? Does this relationship hold for other triangles? Students can investigate these questions, test them, and explain why these relationships are true.

Often students spend the entire year in a geometry class wondering what it has to do with algebra. With the use of the TI-92 students can collect data from geometrical objects, graph the data, and relate the data to symbols. A dynamic geometry tool provides students with an opportunity to reason about a situation in a somewhat different manner.

A number of important points arise from this "functions" use of dynamic geometry. At almost every phase of these investigations students have been asked to justify their conclusions in some way. In particular, they should verify the conjectures that arise from their empirical investigations in developing functional relationships from data. This is the kind of mathematical behavior that we want to encourage–both to explore and to justify. Another important possibility is having students connect parts of the model to the geometry of the context. The use of functional relationships in geometry is much easier with technology and we now have the added benefit of making algebra visible to the students at the same time. This is critical, because if we want students to think in terms of functions then functions must not be absent from the curriculum for a year. It is somewhat like small doses of radiation. Over time it produces effects.

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FOOTNOTE

A variety of dynamic geometry tools are available, including Geometer's Sketchpad, Cabri, and Geometry Inventor. This article will focus on the use of the TI-92 and its dynamic geometry tool.

ABOUT THE AUTHORS

Karen Flanagan is a Graduate Research Trainee and a Ph.D. candidate in Mathematics Education at The Pennsylvania State University. Her address is 113A Chambers Building, University Park, PA 16802.

Ken Kerr is a Graduate Research Trainee and a Ph.D. candidate in Mathematics Education at The Pennsylvania State University. Ken is on leave from teaching mathematics at Glenbrook South High School, Glenview, IL. His current address is 113B Chambers Building, University Park, PA 16802.

Calculators and the Education of Youth

(continued from page 54)

The National Council of Teachers of Mathematics also recommends that every mathematics teacher at every level promote the use of calculators to enhance mathematics instruction by—

modeling the use of calculators in a variety of situations;

using calculators in computation, problem solving, concept development, pattern recognition, data analysis, and graphing;

incorporating the use of calculators in testing mathematical skills and concepts;

keeping current with the state-of-the-art technology appropriate for the grade level being taught;

exploring and developing new ways to use calculators to support instruction and assessment.

The National Council of Teachers of Mathematics further recommends that—

school districts conduct staff development programs that enhance teachers' understanding of the use of appropriate state-of-the-art calculators in the classroom;

teacher preparation institutions develop preservice and in-service programs that use a variety of calculators, including graphing calculators, at all levels of the curriculum;

educators responsible for selecting curriculum materials make choices that reflect and support the use of calculators in the classroom;

publishers, authors, and test and competition writers integrate the use of calculators at all levels of mathematics;

mathematics educators inform students, parent , administrators, and school boards about the research that shows the advantages of including calculators as an everyday tool for the student of mathematics.

Research and experience have clearly demonstrated the potential of calculators to enhance students' learning in mathematics. The cognitive gain in number sense, conceptual development, and visualization can empower and motivate students to engage in true mathematical problem solving at a level previously denied to all but the most talented. The calculator is an essential tool for all students of mathematics.

National Council of Teachers of Mathematics (February, 1991)

MATHEMATICALLY MODELING A TRAFFIC INTERSECTION

Jon Wetherbee
Unami Middle School[1]

The project presented in this chapter entails the development of a mathematical model for traffic flow through an intersection controlled by a traffic light. It involves field study, namely, data collection at an intersection, followed by the use of spreadsheets and computer programming for the development of a mathematical model for traffic flow.

Purpose

This project simulates traffic flow at an intersection's traffic light. The simulation can be used to predict what traffic flow would be if changes were made to the intersection. For example, one might want to know how traffic flow would change if another lane were to be put in or if a new road were to be built to divert some of the traffic that passes through the intersection.

Procedures

The procedures used to build a mathematical model that simulates traffic flow included observation of a particular traffic intersection (steps 1, 2, and 5 that follow), analysis of the data from that observation (steps 3, 4, 6, and 7), use of the results from the data analysis to build a spreadsheet-based simulation of traffic flow through the intersection (steps 8 through 11), and then modification of the simulation to examine changes in traffic flow as a result of proposed changes in the road (steps 12 and 13). The details for each of these steps are given next.[2]

Observation and Data Analysis

1. Time the traffic light from beginning of green to the next beginning of green (the cycle of the light). Take several readings and find their average to ensure accuracy. For example, suppose the average cycle time were $26\frac{2}{3}$ seconds. Then in one hour there would be just over 135 cycles.
2. Over a one hour time period count how many cars approach the light (within some reasonable distance of sight) during each cycle of the light (the time determined in step 1). Since the more days during which one collects data the more accurate the results will be, it is suggested that data be collected on at least three days. Be sure to observe the same intersection at the same time on each day.
3. From the observed data (number of cycles in one hour during which various numbers of cars approached the intersection) find the occurrence percentage for each number of cars (percent of the total number of cycles in one hour during which that number of cars approached the intersection).

See Table 1 for sample data. Since five cars approached during two of the 135 cycles in the hour, the occurrence percentage for five is $\frac{2}{135}$, or 1.48 percent Note that the table is only a partial table of observed results showing only the number of cycles during which five cars approached the intersection and the number of cycles during which six cars approached the intersection and the occurrence percentages for each of those values.

Number of Cars Approaching the Intersection	Frequency of Occurrence (Number of Cycles in One Hour During which This Number of Cars Was Approaching)	Occurrence Percentage
...[a]	...	...
4	...	...
5	2	1.48%
6	5	3.70%
7	...	...
...	...	...

Table 1. Frequency and occurrence percentage for numbers of cycles in one hour during which various numbers of cars approached the intersection.

[a] This is only a partial example. The complete table would include more values.

4. Use the range of integers 0-9999 (the integers in the interval [0,9999]) and let a portion of that range represent each potential number of cars according to the occurrence percentages for the numbers. For example, since the occurrence percentage for five cars is 1.48 percent, use 1.48 percent of the range 0-9999, say 0-147, to correspond to the frequency with which five cars approached the intersection. Similarly, since the occurrence percentage for six cars is 3.7 percent, use 3.7 percent of the range 0-9999, say 148-517, to represent the frequency with which six cars approached the intersection (see Table 2).
5. Using the cycle time in step 1, count how many cars pass through the light during each cycle in one hour. Observe on the same number of days as in step 2.
6. From the observed data (number of cycles in one hour during which various numbers of cars passed through the intersection) find the occurrence percentage for each number of cars (percent of the total number of cycles in one hour during which that number of cars passed through the intersection) (see Table 3).
7. As done in step 4, assign a portion of the range 0-9999 to each potential number of cars according to the occurrence percentages for the numbers (see Table 4)

Number of Cars Approaching the Intersection	Frequency of Occurrence (Number of Cycles in One Hour During which This Number of Cars Was Approaching)	Occurrence Percentage	Portion of Range 0-9999 Corresponding to this Number of Cars Approaching
... [a]	...	...	...
4	...	...	...
5	2	1.48%	0-147
6	5	3.70%	148-517
7	...	...	...
...	...	...	...

Table 2. Frequency, occurrence percentage, and portion of the range 0-9999 representing numbers of cycles in one hour during which various numbers of cars approached the intersection.

[a] This is only a partial example. The complete table would include more values.

Number of Cars Passing through the Intersection	Frequency of Occurrence (Number of Cycles in One Hour During which This Number of Cars Passed through)	Occurrence Percentage
... [a]	...	..
5	...	...
6	2	1.48%
7	2	1.48%
8	...	...
...	...	...

Table 3. Frequency and occurrence percentage for numbers of cycles in one hour during which various numbers of cars passed through the intersection.

[a] This is only a partial example. The complete table would include more values.

Number of Cars Passing through the Intersection	Frequency of Occurrence (Number of Cycles in One Hour During which This Number of Cars Passed through)	Occurrence Percentage	Portion of Range 0-9999 Corresponding to this Number of Cars Passing through
... [a]	...	...	...
5	...	...	...
6	2	1.48%	0-147
7	2	1.48%	148-295
8	...	...	...
...	...	...	...

Table 4. Frequency, occurrence percentage, and portion of the range 0-9999 representing numbers of cycles in one hour during which various numbers of cars passed through the intersection.

[a] This is only a partial example. The complete table would include more values.

Building a Spreadsheet-based Simulation

8. Using a spreadsheet, have a computer generate random numbers from the range 0-9999 that can be used to simulate the number of cars approaching the intersection during cycles of the traffic light.
9. Program the computer (this can be done in the spreadsheet Microsoft Excel using If...Then statements) to report that if the random number falls into a particular range of integers then the number of cars approaching the intersection represented by that range would be used for that cycle. (In Microsoft Excel this can be done using If...Then statements.) Note that some spreadsheets have limitations on the number of If...Then statements per cell and that you may need to test whether a random number falls, for example, into one of 15 or more ranges. To avoid this problem it is possible to use several cells for successive If...Then tests. One can run through seven (or as many as your program will allow) If...Then statements in one cell and tell the computer to report that if none of them are true, then put in -1 (or some other illogical value). Then have the If...Then statements in the next cell refer back to the first cell, saying that if the first cell does not equal -1, then the value in the second cell equals the value in the first cell, and if the first cell does equal -1 then go on to the next set of If...Then statements in the second cell. If all of the If...Then tests in the third cell are not true, then the value in the third cell is -1 and the final result would be the value corresponding to that part of the 0-9999 range not included in any of the If...Then statements. The results of random number generation and such successive If...Then tests on those numbers are illustrated in Table 5. In a given row of Table 5 the cells of columns three through five display the results of three such sets of If...Then tests. Note that the final result of nine for cycle eight corresponds to the portion of the 0-9999 range not included in the three sets of If...Then statements.
10. Repeat step 9, generating another set of random numbers used to simulate the number of cars passing through, rather than approaching, the intersection during each cycle. If necessary, use sets of If...Then statements to find the appropriate portion of the range 0-9999 for each random number generated.
11. For each cycle use the final value for the number of cars passing through the intersection and the final value for the number of cars approaching the intersection to produce the number of cars remaining at the intersection (subtract the number of cars passing through the intersection from the number of cars approaching the intersection and add to the previous number of cars remaining) (see Table 6).

Cycle	Random Number Generated to Simulate Number of Cars Approaching the Intersection	Result of Set of If... Then tests	Result of Set of If... Then tests	Result of Set of If... Then tests	Final Result for Number of Cars Approaching
1	7532	-1	19	19	19
2	6413	-1	18	18	18
3	2726	-1	-1	13	13
4	2370	-1	-1	13	13
5	9162	22	22	22	22
6	6633	-1	18	18	18
7	9027	22	22	22	22
8	508	-1	-1	-1	9
... [a]	...	...	...	...	...

Table 5. Random numbers and results of If...Then tests determining the number of cars approaching the intersection.

[a] This is only a partial example. The complete table would include more cycles.

Cycle	Random Number Generated to Simulate Number of Cars Approaching the Intersection	Random Number Generated to Simulate Number of Cars Passing through the Intersection	Number of Cars Approaching	Number of Cars Passing through	Number of Cars Remaining
1	7532	1609	19	10	9
2	6413	196	18	7	20
3	2726	872	13	10	23
4	2370	734	13	9	27
5	9162	6249	22	14	35
6	6633	2232	18	11	42
7	9027	6363	22	14	50
8	508	272	9	7	52
9	1650	2321	12	11	53
10	1505	947	11	10	54
11	3916	9288	15	18	51
12	7809	3298	19	12	58
13	8877	8445	22	16	64
... [a]	...	...	...	...	...

Table 6. Model, built from an actual study during rush hour traffic, predicting number of cars remaining at an intersection.

[a] This is only a portion of the simulation for demonstration purposes. A complete hour would include more cycles.

Modifications of the Simulation

After simulating traffic flow on one lane of a two-lane road (one in each direction), one might want to know what the traffic conditions would be like if another lane were built or if another road were built and some of the traffic used the new road.

12. Find the number of cars that would not be in the current lane if another lane were built. To do this, divide the number of cars approaching during each cycle of the simulation by two. If a decimal value results for the number of cars approaching the intersection, the number should be rounded to a whole number. Table 7 displays modification of the results from Table 6 reflecting the building of a second lane.

Cycle	Random Number Used to Simulate Number of Cars Approaching the Intersection	Random Number Used to Simulate Number of Cars Passing through the Intersection	Number of Cars Approaching Intersection as Determined by Random Number in Column 2	(Number of Cars Approaching Intersection) ÷ 2	Number of Cars Passing Through Intersection as Determined by Random Number in Column 3	Cars Remaining Traffic Light (Column 5 minus Column 6)
1	7532	1609	19	10 [a]	10	0
2	6413	196	18	9	7	2
3	2726	872	13	7	10	0
4	2370	734	13	7	9	0
5	9162	6249	22	11	14	0
6	6633	2232	18	9	11	0
7	9027	6363	22	11	14	0
8	508	272	9	5	7	0
9	1650	2321	12	6	11	0
10	1505	947	11	6	10	0
11	3916	9288	15	8	18	0
12	7809	3298	19	10	12	0
13	8877	8445	22	11	16	0
... [b]	...	...	...	...	...	...

Table 7. Modified simulation predicting traffic flow if an additional lane were built.

[a] Values in this column are rounded.
[b] This is only a partial example. A complete table would include more cycles.

13. To simulate traffic conditions on the current road if another road were built, estimate what percent of the cars on the current road would use the new road. Then use that percent to decrease the number of cars approaching the intersection. Table 8 shows the traffic conditions with another lane added to the road, assuming that 18 percent of the cars on the current road would use the new road.

Cycle	Random Number Used to Simulate Number of Cars Approaching the Intersection	Random Number Used to Simulate Number of Cars Passing through the Intersection	Number of Cars Approaching Intersection as Determined by Random Number in Column 2	Number of Cars Approaching Intersection if Bypass Were Built (Column 4 minus 18% of Column 4, of 82% of Column 4)	Number of Cars Passing Through Intersection as Determined by Random Number in Column 3	Cars Remaining Traffic Light (Column 5 minus Column 6)
1	7532	1609	19	16 [a]	10	6
2	6413	196	18	15	7	14
3	2726	872	13	11	10	15
4	2370	734	13	11	9	17
5	9162	6249	22	18	14	21
6	6633	2232	18	15	11	25
7	9027	6363	22	18	14	29
8	508	272	9	7	7	29
9	1650	2321	12	10	11	28
10	1505	947	11	9	10	27
11	3916	9288	15	12	18	21
12	7809	3298	19	16	12	25
13	8877	8445	22	18	16	27
... [b]	...	...	...	...	...	...

Table 8 Modified simulation predicting traffic flow if a new road were built causing an 18 percent decrease in traffic on the original road

[a] Values in this column are rounded

[b] This is only a partial example. A complete table would include more cycles.

Results from the original simulation and the two modifications also can be graphed. A sample graph is given in Figure 1. For comparison one might choose to graph the average number of cars remaining for several simulations of each condition on the same axes.

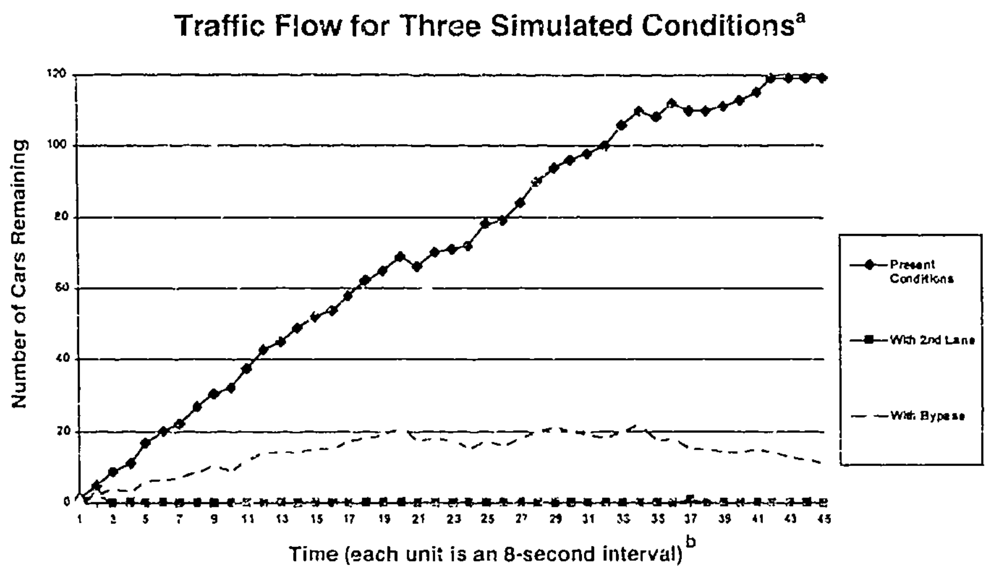

Figure 1. Predicted traffic flow for current road, current road with second lane, and current road with bypass.

[a] Values represent averages over several simulations.

[b] Each 80-second time interval represents 3 cycles.

FOOTNOTES

At the time he submitted this article, Jon Wetherbee was a ninth-grade student attending Unami Middle School, Chalfont, Pennsylvania, in the Central Bucks School District. This project was his entry into the Benjamin Banneker Science Fair. Jon worked with two teachers, Mr. William Rissinger, a science teacher, and Mr. Lee Ellmaker, a mathematics teacher.

[*Editors' Note*: Jon Wetherbee's account of his project provides a look, through the eyes of a student, at a middle school project that technology makes possible. This project illustrates one type of mathematical modeling activity in which teachers might engage their students. It is left to the reader to envision the format in which a project such as this might be assigned, the particular questions a teacher might raise while students are collecting and analyzing data and building models, and the desired mathematical outcomes from students' completion of such a project.]

ABOUT THE AUTHOR

Jon Wetherbee currently is a student at Central Bucks West High School in Doylestown, Pennsylvania. His address is 19 Farber Drive, Chalfont, PA 18914.

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